Unitary synthesis and unclonable cryptography

John Bostanci

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I’m going to tell you about two big areas related to this: the unitary synthesis problem, and unclonable cryptography.

Quantum preliminaries

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An n-qubit state is a norm 1, 2^n dimensional vector of complex numbers.

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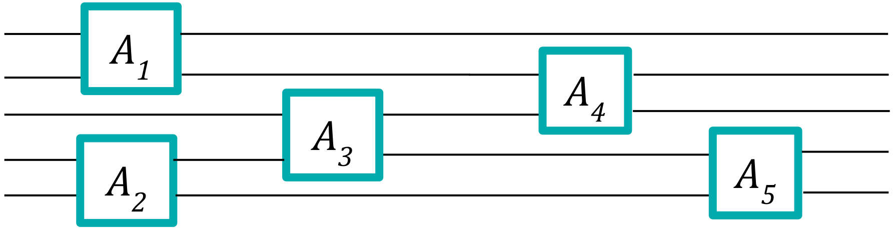
An n-qubit unitary is a $2^n \times 2^n$ norm preserving matrix.

$$\|U|\psi\rangle\| = \||\psi\rangle\|$$

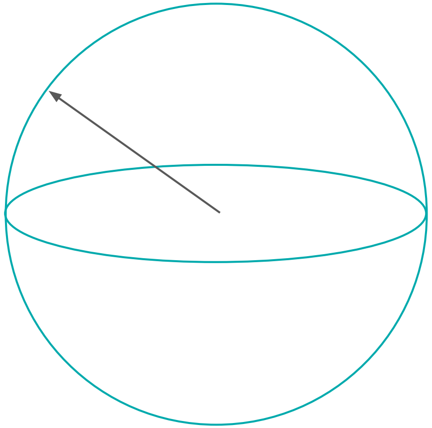
Quantum preliminaries

Efficient quantum computation is a poly(n) sized quantum circuit consisting of a sequence of two-qubit unitary gates.



Quantum preliminaries

A “random” state refers to a Haar random vector from the unit ball in $\mathbb{C}^{2^n}$.



The unitary synthesis problem

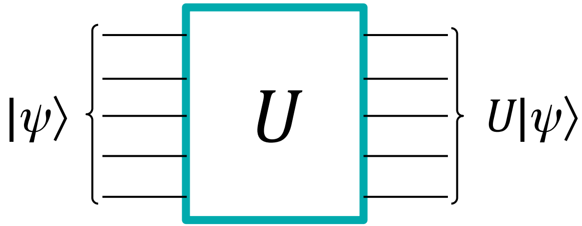
Motivating question:

Can we relate the complexity of “quantum problems” to the complexity of “classical problems”?

Classical problems can typically be reduced to the task of computing some function,

$$f : \{0, 1\}^* \rightarrow \{0, 1\}^*$$

The most general transformation that can be implemented on a quantum computer is called a unitary transformation.



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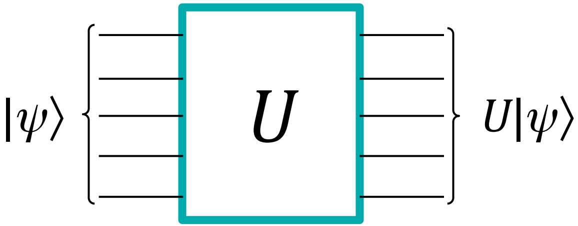

Question [AK'07]: Can we efficiently reduce every unitary to ***some*** function f?

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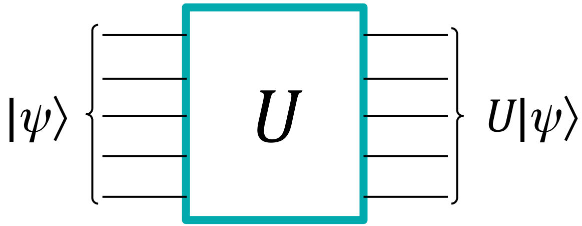
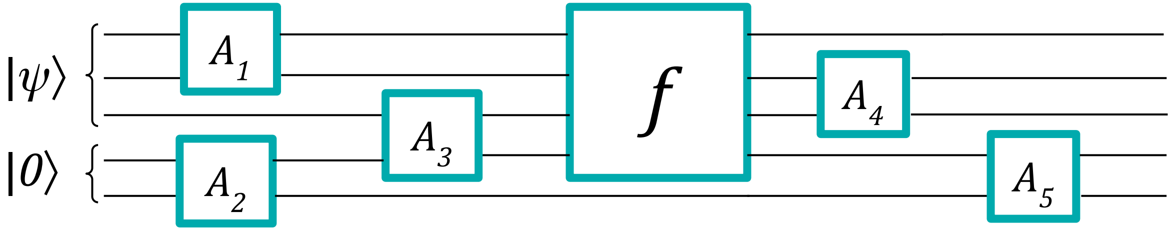


Question [AK'07]: Can we efficiently reduce every unitary to ***some*** function f?

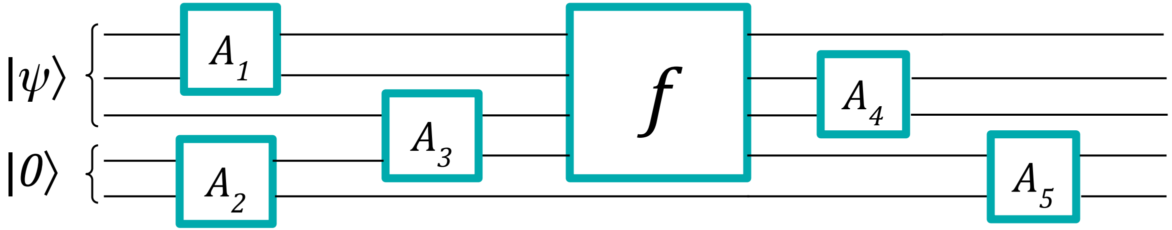
Query algorithms

A query algorithm A^f is a sequence of unitaries, with superposition queries to the function f in between.



Unitary synthesis

Is there a universal, efficient, query algorithm A, such that for any unitary U, there exists a function f such that A^f implements U?



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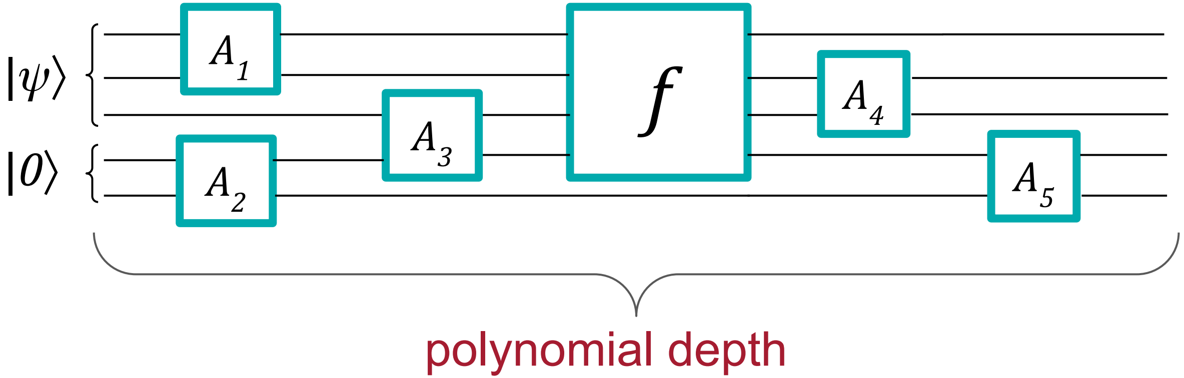

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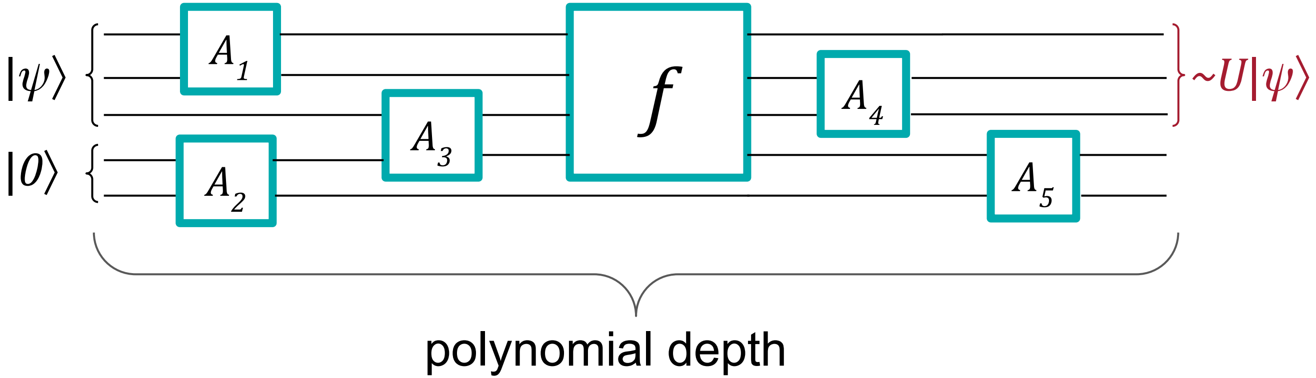

Unitary synthesis and cryptography

Both directions of the unitary synthesis problem seem to have connections to quantum cryptography.

Single-copy pseudo-random states

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- (Stretch) The number of qubits of $|\psi_{n,k}\rangle$ is greater than n.
- (Pseudo-randomness) For all efficient adversaries A, the following holds

$$|\mathrm{Pr}_{\mathrm{k}}[A(|\psi_{n,k}\rangle) \text{ accepts}] - \mathrm{Pr}_{\psi}[A(|\psi\rangle \text{ accepts})]| = \mathrm{negl(n)}$$

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- An NP oracle?
- A PP oracle?
- A RE oracle?

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If the unitary synthesis problem is resolved in the negative, then there exists constructions relative to oracles that can not be broken by an efficient adversary given oracle access to any classical problem.

Unitary synthesis: what is known?

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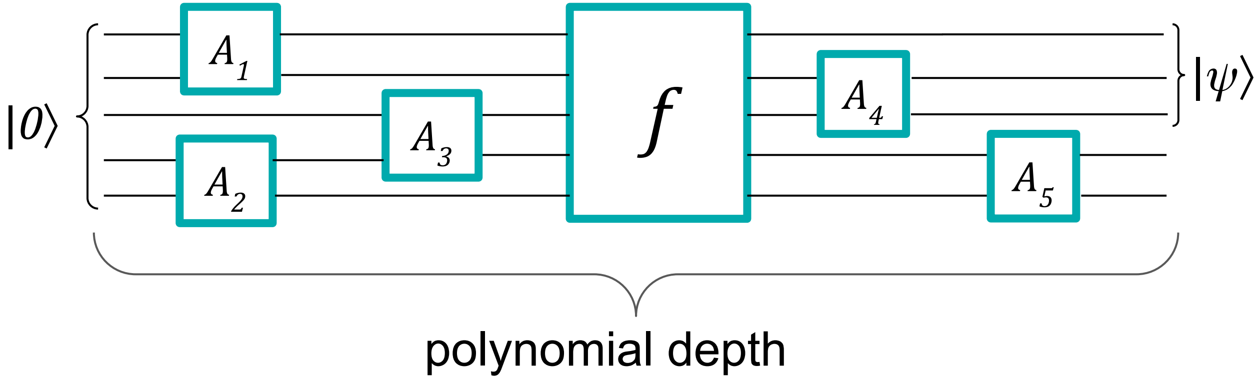
Lower bound: You need at least 2 queries to the function [LMW’24].

Upper bound: You only need $2^{n/2}$ queries to the function [Rosenthal’21].

Related problem: state synthesis

In state synthesis, we want to design a fixed query algorithm that, on input 0^n, queries a function to prepare a state $|\psi\rangle$.



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We know how to do state synthesis efficiently only using a single query to a function [Rosenthal'23, INNRY'22].

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I personally think you shouldn't be able to synthesize unitaries, but proving this is challenging, so I can't say for sure how the proof would go.

Either way will be exciting and tell us about the interplay between quantum and classical complexity theory!

Unclonable cryptography

No-cloning

The no-cloning theorem says that there is no algorithm that clones an unknown quantum state.

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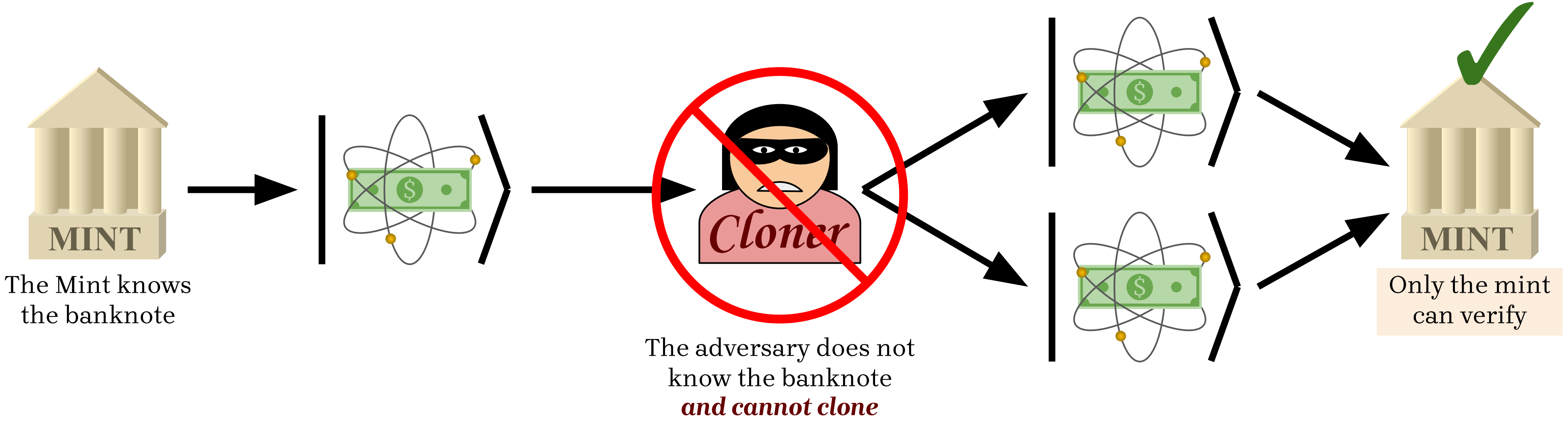
In the 1970’s, Stephen Wiesner imagined a use case: Money that no one can copy.

Private-key quantm money [Wiesner'83]



Credits: Barak Nehoran

Private-key quantum money [Wiesner'83]

A private-key quantum money scheme consists of the following algorithms:

- $\mathrm{Gen}(1^{\mathrm{n}}) \rightarrow (sk, s, |\psi_s\rangle)$
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The scheme is secure if, for all adversaries A, the following holds.

$$\Pr\left[\mathrm{Ver}(\mathrm{sk}, \mathrm{s}, |\psi_1\rangle) \text{ and } \mathrm{Ver}(\mathrm{sk}, \mathrm{s}, |\psi_2\rangle) \;\middle|\; \begin{array}{c} (sk, s, |\psi_s\rangle) \leftarrow \mathrm{Gen}(1^{\mathrm{n}}), \\ (|\psi_1\rangle, |\psi_2\rangle) \leftarrow \mathrm{A}(s, |\psi_s\rangle) \end{array}\right] = \mathrm{negl}(\mathrm{n})$$

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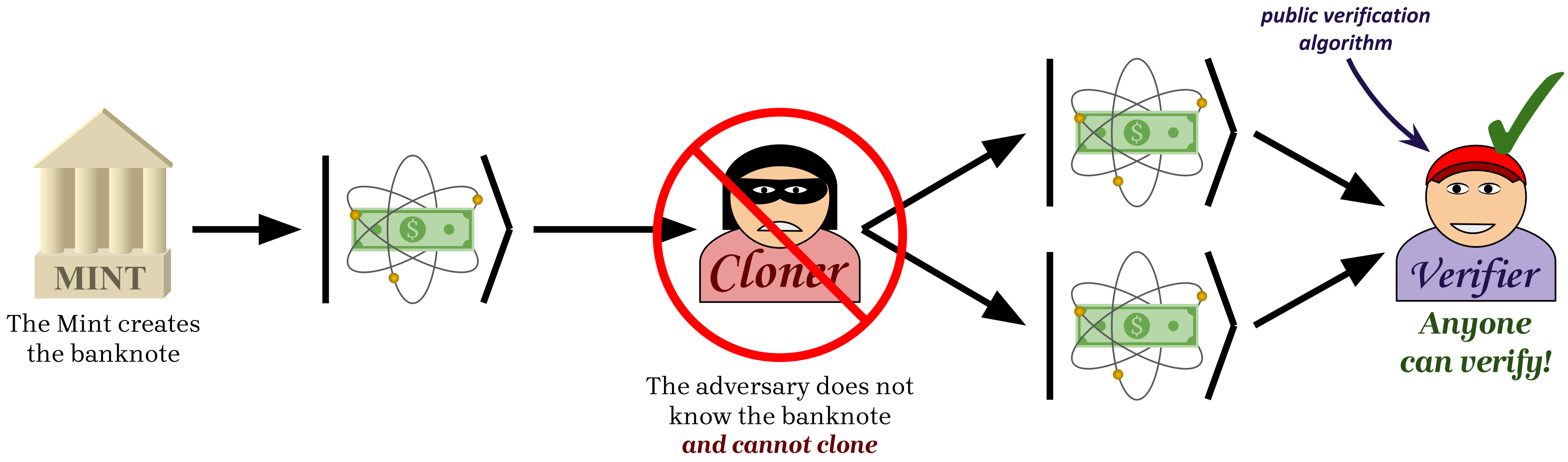
This inspired "public-key" quantum money.

Public-key quantum money [Aaronson'09]



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Quantum lightning [Zhandry'17]

A quantum lightning scheme consists of the following algorithms:

- $\mathrm{KeyGen}(1^{\mathrm{n}}) \rightarrow (sk, pk)$
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$$\Pr\left[\mathrm{Ver}(\mathrm{sk}, \mathrm{s}, |\psi_1\rangle) \text{ and } \mathrm{Ver}(\mathrm{sk}, \mathrm{s}, |\psi_2\rangle) \,\middle|\, (\mathrm{s}, |\psi_1\rangle, |\psi_2\rangle) \leftarrow \mathrm{A}(1^n)\right] = \mathrm{negl}(\mathrm{n})$$

Public-key quantum money

Quantum money and lightning have been notoriously difficult to construct!

Merely conjectured

[Aaronson'09]: random stabilizer states

broken by [Lutomirski-Aaronson-Farhi-Gosset-Hassidim-Kelner-Shor'10]

[Farhi-Gosset-Hassidim-Lutomirski-Shor'10]: knots

[Aaronson-Christiano'12]: polynomials hiding subspaces

broken by [Pena-Faugère-Perret'14, Christiano-Sattath'16]

[Zhandry'19]: quadratic systems of equations

broken by [Roberts'21]

[Kane'18, Kane-Sharif-Silverberg'21]: quaternion algebras

[Khesin-Lu-Shor'22]: lattices

broken by [Liu-Montgomery-Zhandry'23]

Black box / oracle proof

[Aaronson'09]: quantum oracle

[Aaronson-Christiano'12]: classical hidden subspaces oracle

[Kane'18, Kane-Sharif-Silverberg'21]: Commuting unitaries

[Liu-Montgomery-Zhandry'23]: Walkable invariants

[Zhandry'23]: Abelian group actions

Black-box proofs may break in plain model

Reduction from a widely-studied assumption

[Zhandry'19]: indistinguishability obfuscation

Very strong assumption! quantum security unclear

No known security proof for quantum lightning in the plain model

Chart adapted from [Zhandry'23, Nehoran'24]

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- Act as signature keys (Tokenized signatures [BS'16, Shmueli'22])
- Allow us to execute arbitrary functions (Copy-protected software [Aaronson'09, AL'21, ALLZZ'21, BJLPS'21, CMP'24])

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Constructing public-key quantum money in the plain model and proving security from a "well-founded" assumption is a major open problem!