

A General Quantum Duality for Representations of Groups

with Applications to Quantum Money, Lightning, and Fire

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Abstract

Aaronson, Atia, and Susskind [AAS20] established that efficiently mapping between quantum states $|\psi\rangle$ and $|\phi\rangle$ is computationally equivalent to distinguishing their superpositions $\frac{1}{\sqrt{2}}(|\psi\rangle + |\phi\rangle)$ and $\frac{1}{\sqrt{2}}(|\psi\rangle - |\phi\rangle)$. We generalize this insight into a broader duality principle in quantum computation, wherein manipulating quantum states in one basis is equivalent to extracting their value in a complementary basis. In its most general form, this duality principle states that for a given group, the ability to implement a unitary representation of the group is computationally equivalent to the ability to perform a Fourier subspace extraction from the invariant subspaces corresponding to its irreducible representations.

Building on our duality principle, we present the following applications:

- Quantum money, which captures quantum states that are verifiable but unclonable, and its stronger variant, quantum lightning, have long resisted constructions based on concrete cryptographic assumptions. While (public-key) quantum money has been constructed from indistinguishability obfuscation (iO)—an assumption widely considered too strong—quantum lightning has not been constructed from any such assumptions, with previous attempts based on assumptions that were later broken. We present the first construction of quantum lightning with a rigorous security proof, grounded in a plausible and well-founded cryptographic assumption. We extend Zhandry’s construction from Abelian group actions [Zha24] to non-Abelian group actions, and eliminate Zhandry’s reliance on a black-box model for justifying security. Instead, we prove a direct reduction to a computational assumption – the pre-action security of cryptographic group actions. We show how these group actions can be realized with various instantiations, including with the group actions of the symmetric group implicit in the McEliece cryptosystem.
- We provide an alternative quantum money and lightning construction from one-way homomorphisms, showing that security holds under specific conditions on the homomorphism. Notably, our scheme exhibits the remarkable property that four distinct security notions—quantum lightning security, security against both worst-case cloning and average-case cloning, and security against preparing a specific canonical state—are all equivalent.
- Quantum fire captures the notion of a samplable distribution on quantum states that are efficiently clonable, but not efficiently telegraphable, meaning they cannot be efficiently encoded as classical information. These states can be spread like fire, provided they are kept alive quantumly and do not decohere. The only previously known construction relied on a unitary quantum oracle, whereas we present the first candidate construction of quantum fire in the plain model.

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1 Introduction

Let $|\psi_0\rangle, |\psi_1\rangle$ be two orthogonal quantum states, and let $|\phi_+\rangle$ be proportional to $|\psi_0\rangle + |\psi_1\rangle$ and $|\phi_-\rangle$ be proportional to $|\psi_0\rangle - |\psi_1\rangle$. The *Swap Complexity* of $|\psi_0\rangle$ and $|\psi_1\rangle$ is the size of the smallest circuit that maps $|\psi_0\rangle$ to $|\psi_1\rangle$ and vice versa. Meanwhile, the *Distinguishing Complexity* of $|\phi_+\rangle$ and $|\phi_-\rangle$ is the size of the smallest circuit that accepts $|\phi_+\rangle$ and rejects $|\phi_-\rangle$. A fundamental result of Aaronson, Atia, and Susskind [AAS20] establishes that the swap complexity of $|\psi_0\rangle$ and $|\psi_1\rangle$ is essentially equivalent to the distinguishing complexity of $|\phi_+\rangle$ and $|\phi_-\rangle$. This duality principle, known as the “AAS duality”, has emerged as a simple yet powerful tool in quantum complexity theory and cryptography.

In this work, we ask: *Can the AAS equivalence be extended to the more general context of many quantum states and multidimensional subspaces?* We give an affirmative answer to this question. First, we extend the notion of the swap complexity to a notion of “representation complexity”: given a subspace, V , spanned by states $|\psi_1\rangle, \dots, |\psi_k\rangle$, and a (potentially non-Abelian) group G , a *representation* of G on the subspace V is a homomorphism from G to the unitaries acting on the subspace (or, roughly, it is a collection of unitaries $\{U_g\}_{g \in G}$ acting on V which satisfies the group operations of G).¹ Its *Representation Complexity* is the size of the smallest circuit that implements the representation, that is, by mapping

$$|g\rangle \otimes |\psi_i\rangle \mapsto |g\rangle \otimes U_g |\psi_i\rangle .$$

When restricting to groups that have an efficient quantum Fourier transform (including all Abelian groups, all constant-sized or polynomial-sized non-Abelian groups, and several important exponential-sized non-Abelian groups), we show that the representation complexity is essentially equivalent to the complexity of implementing a *Fourier subspace extraction*, or in other words, performing a partial measurement of the invariant subspaces preserved by the representation (i.e., its irreducible representation subspaces) and extracting the quantum state encoded in each such subspace (see Section 1.2 for more discussion on subspace extraction). For Abelian groups, this simplifies to a full projective measurement, and in particular, for the swapping representation of AAS, this is a measurement between $|\phi_+\rangle$ and $|\phi_-\rangle$. Thus the AAS duality is recovered by setting $G = \mathbb{Z}_2$. We additionally prove an *approximate* notion of this duality, where the circuit only has to approximately map between states.²

1.1 Applications to Cryptography

In cryptography, the AAS duality has proven quite fruitful. Cryptographic security properties come in two types: “search” type properties which stipulate the hardness of computing a specific unknown quantity, and “decision” type properties which stipulate the hardness of distinguishing between two distributions. The AAS duality has played a crucial role in establishing the equivalence between certain search-type and decision-type properties, leading to a number of significant results [Yan22, HMY23, KMNY24, MW24, HKNY24, MYY24].

We show that our new duality theorem is useful for cryptography beyond the AAS setting, by giving novel results for quantum money.

Quantum Money from Group Actions. Quantum money [Wie83] uses the no-cloning principle to generate unforgeable banknotes. These banknotes are quantum states that can be verified

¹For instance, in [AAS20], the representation of $G = \mathbb{Z}_2$ on the subspace spanned by $|\psi_0\rangle$ and $|\psi_1\rangle$ maps the sole non-identity element of \mathbb{Z}_2 to the unitary swapping $|\psi_0\rangle$ for $|\psi_1\rangle$.

²While our approximate duality theorem works for all groups, it achieves weaker error bounds for general groups.

but cannot be cloned. A central problem has been to construct quantum money that can be *publicly verified* by anyone, and yet only the mint can create new banknotes. This is called public-key quantum money [Aar09].³ Quantum *lightning* posits a stronger security notion for public-key quantum money, with a collision-resistance property that ensures that even the mint can only ever create one copy of each banknote [Zha21].

A long-standing challenge for public-key quantum money is to derive security from concrete computational assumptions (and in particular, assumptions that do not bake the unclonability of the banknotes directly into the assumption). The only prior scheme with such a proof is an instantiation of [AC12] using indistinguishability obfuscation (iO), as suggested by [BDS23] and proved in [Zha21]. However, iO is a powerful cryptographic tool whose quantum security is still uncertain. Moreover, no existing unbroken scheme has been shown to have such a security proof for the stronger security notion of quantum lightning.

Recently, [Zha24] gave a plausible construction of quantum money and quantum lightning from Abelian group actions. A group action consists of a group G , a set X , and a binary operation $*$: $G \times X \rightarrow X$, denoted $g * x = y$. This operation respects the group structure: $g * (h * x) = (gh) * x$. An Abelian group action is a group action where G is Abelian.⁴ Unfortunately, the security of the scheme of [Zha24] requires both a computational assumption *and* an idealized modeling of group actions as a black box.

Using our duality principle, we show how to generalize this construction to work with *non-Abelian* group actions. This shift is not merely a superficial adjustment—it significantly improves on the framework in two critical ways:

1. It allows us to prove the hardness of our quantum money and lightning scheme in the *standard model*, using only a concrete assumption on the group action. This assumption also identifies an interesting potential source of hardness for non-Abelian group actions. Very roughly, for non-Abelian groups, in addition to a group action $g * (h * x) = (gh) * x$, we can also define a “pre-action” $g \circ (h * x) = (hg^{-1}) * x$, or more generally a “bi-action” $(g_0, g_1) \circledast (h * x) = (g_0 h g_1^{-1}) * x$. Our assumption states that it is hard via a quantum query to distinguish a random action from a random bi-action. Importantly, this problem only makes sense for non-Abelian group actions, as actions and pre-actions are identical in the Abelian case. Thus, the quantum money result requires us to use the full power of our non-Abelian generalization of the duality.
2. The shift to non-Abelian groups opens up the possibility for potentially more varied instantiations of the group actions. In particular, we explain how to instantiate our quantum money scheme on (a significant generalization of) the symmetric group action implicit in the McEliece cryptosystem [McE78].

Theorem 1.1 (informal). *There is a public-key quantum money and quantum lightning scheme for any (non-Abelian) cryptographic group action, such that the money/lightning scheme is secure if the group action is preaction-secure.*

To the best of our knowledge, this represents the first (unbroken) quantum lightning scheme with a standard-model security proof based on a computational assumption that does inherently include unclonability.

³When it is otherwise clear from context, we will refer to public key quantum money as simply “quantum money”.

⁴Abelian groups are those for which all the elements commute: $gh = hg \ \forall g, h \in G$.

Quantum Money from One-way Homomorphisms. A one-way (group) homomorphism is a function, $f(h)$, that is group homomorphic⁵ and efficiently computable, but computationally intractable to invert.⁶ A one-way homomorphism can be seen as an instance of a group action, with the domain group acting on the codomain as $g * f(h) = f(gh)$. However, unlike in the previous case above, the preactions for this action (i.e., $g \circ f(h) = f(hg^{-1})$) are as efficiently computable as the action itself, so security cannot be shown as before. Nevertheless, we give sufficient conditions on the one-way homomorphism such that the resulting quantum lightning scheme is secure.

We note that unlike our construction above from group actions that are pre-action secure—for which we give concrete instantiations that can be implemented in practice—we do not know if any instantiations of homomorphic functions satisfy these security conditions. But we observe that a one-way group homomorphism is essentially a group action where the computational Diffie-Hellman (CDH) problem is *easy* but yet discrete logarithms are still *hard*. While CDH is quantumly equivalent to discrete logarithms for Abelian groups [MZ22], this equivalence does not seem to follow for non-Abelian groups. Strangely, it is a hypothetical security *failure* for group actions which gives rise to plausible instantiations for quantum lightning and quantum fire (see more on the construction of quantum fire below).

We concede the disadvantage of this construction as compared to the concrete one above from preaction security, but we note that it has some unique properties that the other does not. Specifically, by leveraging our duality principle we are able to prove the remarkable fact that four distinct quantum money security notions—namely, the collision-resistance of quantum lightning security, the hardness of both worst-case cloning and average-case cloning, and the hardness of preparing the uniform superposition over the image of the homomorphism—are all identical. Thus for any particular instantiation of the one-way homomorphism, it is sufficient to prove any one of these security notions in order to get the other three.

Quantum Fire. Quantum fire refers to a collection of efficiently samplable quantum states that can be efficiently cloned, but cannot be efficiently telegraphed.⁷ That is, despite the ability to make an unbounded number of copies of a quantum fire state, there is no way to efficiently encode it as classical information from which it can later be recovered. Much like a flame can be easily spread from a single source as long as it is kept alive, quantum fire can be cloned from a single quantum state as long as it is kept coherently in quantum storage.

The concept of quantum fire was first introduced in the work of Nehoran and Zhandry [NZ23], where it was shown to be essential for solving the key exfiltration problem. However, it was not formally defined or named in that work. [NZ23] provided a secure construction of quantum fire relative to a unitary quantum oracle, but this oracle construction relied on an inherently inefficient computation and baked clonability into the oracle itself. Consequently, it does not provide a pathway for instantiation in the standard model. It has not even been clear if any

⁵That is, it is a homomorphism between two groups G and H , such that $f(gh) = f(g) \cdot f(h)$ for all $g, h \in G$.

⁶Note that Shor’s algorithm [Sho94] allows efficiently inverting group homomorphisms when the domain and codomain groups are Abelian. Thus, these results inherently require non-Abelian groups, and hence our generalized duality.

⁷Note that while the no-cloning theorem prohibits cloning *general* quantum states, this prohibition does not apply to quantum states chosen from an orthogonal set. The same applies to the no-telegraphing theorem, which prohibits sending *general* quantum states through a classical channel without pre-shared entanglement. States from an orthogonal set can clearly be telegraphed by measuring them in this basis and later recreating them accordingly. Such states can be cloned in a similar fashion. In other words, any states chosen from an orthogonal set can be both cloned and telegraphed *information-theoretically*, but these tasks are not necessarily both efficient. In fact, it was shown in [NZ23] that there are likely to be state families where cloning is efficient and yet telegraphing is not. Quantum fire is the cryptographic primitive that samples such states efficiently.

classical oracle could allow efficient cloning of quantum states that are inherently quantum (and thus not telegraphable).

Inspired by the duality principle, we give a plausible candidate construction of quantum fire relative to a one-way group homomorphism. Remarkably, despite the similarity to the construction of quantum lightning from group homomorphisms, where the states are *unclonable*, the states in this scheme are inherently *clonable*, and efficiently so. Nevertheless, there is no apparent way to telegraph the states efficiently. Moreover, it is straightforward to define a classical oracle that gives a candidate one-way group homomorphism. Thus, we obtain a candidate construction of quantum fire with conjectured security relative to a classical oracle, improving upon the unitary oracle construction of [NZ23].⁸

1.2 The Duality

Fourier Subspace Extraction. A major stepping stone towards our duality theorem is the idea of a Fourier subspace extraction. Every group representation preserves some set of invariant subspaces $\{W_\lambda\}_{\lambda \in [n]}$.⁹ A *course* Fourier measurement¹⁰ of the representation is, roughly, a projection onto these subspaces. We get a classical label λ indicating the subspace we have projected onto, as well as a collapsed state, $|\psi\rangle$, within the subspace W_λ . A *fine* Fourier measurement further measures within each of those subspaces, in a basis that depends on the algorithm. For instance, if $\{|\psi_j^\lambda\rangle\}_{j \in \dim(W_\lambda)}$ is a basis for W_λ , we get outcomes λ and j , and collapse our state to $|\psi_j^\lambda\rangle$.¹¹ In either case, the state after the measurement is still within the subspace.

In some applications, we care about the *coherent* information encoded within each subspace. That is, it is not enough to know which collapsed state $|\psi_j^\lambda\rangle$ we received. We want to have, in our hands, the coherent superposition that appeared in the subspace. That is, if the original state was $\sum_{j \in [\dim(W_\lambda)]} \alpha_j |\psi_j^\lambda\rangle$, we want to *extract* the full superposition $\sum_j \alpha_j |j\rangle$. This transformation, which we call a *subspace extraction*, extracts the full state coherently from the subspace.¹²

If implemented naïvely, Fourier measurements do not suffice for this task. They either do not recover the information about where the state was *within* each subspace (in the case of course Fourier measurement), or they recover it in a collapsed form (in the fine case). In our work, we consider the stronger notion of a “*Fourier subspace extraction*”, an operation that measures the subspace and *coherently recovers* the encoded state.

⁸Note that, as observed in [NZ23], an unconditional security proof relative to such a classical oracle would require proving a classical oracle separation between the complexity classes QMA and QCMA, a major open problem of Aharonov and Naveh [AN02], which, despite recent progress, has evaded resolution.

⁹That is, these subspaces are invariant under all of the unitaries U_g corresponding to each group element $g \in G$. In some cases, the only invariant subspace may be the full Hilbert space, in which case we say that it is *irreducible*, but this is not generically the case. We consider here only invariant subspaces which are irreducible, and do not break down further into smaller invariant subspaces.

¹⁰Often called weak Fourier sampling in many contexts

¹¹To simplify the notation, we assume here that there is no multiplicity, or degeneracy, in the irreducible representations. We will see later how to handle multiplicity.

¹²Note that such extraction is not generally an efficient transformation for arbitrary subspaces.

Duality. We show that the implementations of representations and the implementations of their Fourier subspace extractions are essentially computationally dual to each other.

Theorem 1.2 (Duality, *informal*). *Let G be a group¹³ and let $\mathcal{F} : G \rightarrow U(\mathcal{H})$ be a representation of G . Then the following are equivalent:*

- \mathcal{F} has an efficient implementation, i.e. $|g\rangle \otimes |\psi\rangle \mapsto |g\rangle \otimes \mathcal{F}(g)|\psi\rangle$.
- \mathcal{F} has an efficient Fourier subspace extraction, i.e. $|\psi_{i,j}^\lambda\rangle \mapsto |\phi_i^\lambda\rangle |\lambda, j\rangle$.

Further Discussion of Fourier Subspace Extraction. In the above discussion, we have glossed over the possibility of *degeneracy*, in which the representation acts identically on several different invariant subspaces $W_1^\lambda, W_2^\lambda, \dots, W_m^\lambda$. Such subspaces are degenerate in the sense that a coarse Fourier measurement produces the same outcome, λ , on all of them. Thus we have an additional index, i , that runs over this multiplicity of λ .

We write a Fourier subspace extraction as an isometry $|\psi_{i,j}^\lambda\rangle \mapsto |\phi_i^\lambda\rangle |\lambda, j\rangle$, where for each λ and i , the states $\{|\psi_{i,j}^\lambda\rangle\}_j$ are a basis for the subspace W_i^λ , and the state $|\phi_i^\lambda\rangle$ is an arbitrary “junk” state that is left behind after measuring λ and extracting j .

In order for it to be an *extraction* of j , rather than a measurement of j , it is crucial that this leftover state has no dependence on j . Consider a superposition $\sum_{j \in [\dim(W_i^\lambda)]} \alpha_j |\psi_{i,j}^\lambda\rangle$ over the subspace W_i^λ . Performing this isometry yields $\sum_j \alpha_j |\phi_i^\lambda\rangle |\lambda, j\rangle = |\phi_i^\lambda\rangle |\lambda\rangle \otimes \sum_j \alpha_j |j\rangle$, which extracts the original superposition into a quantum state on the last register with those exact amplitudes. If the leftover junk state had depended on j , for instance if we instead had $|\psi_{i,j}^\lambda\rangle \mapsto |\phi_{i,j}^\lambda\rangle |\lambda, j\rangle$, then this would not extract the state properly. We would instead get $\sum_j \alpha_j |\phi_{i,j}^\lambda\rangle |\lambda, j\rangle$, where the last register is still entangled with the rest of the state, and thus has not been extracted. This is the difference between a *measurement* of j and an *extraction* of j .

We observe that since these leftover junk states $|\phi_i^\lambda\rangle$ are independent of j —that is, they do not depend on which state we started from within the subspace W_i^λ —we can see that these states are instead characteristic of the subspace W_i^λ itself. That is, the Fourier subspace extraction collapses each subspace W_i^λ to a single distinct quantum state $|\phi_i^\lambda\rangle$, which we therefore call the “archetype” states of these subspaces. Despite appearing to be just the “junk” that is left behind during the Fourier subspace extraction, these archetype states are in fact quite useful.

For instance, the existence of these archetype states allows us to use a swap test to distinguish whether two quantum states are in the same subspace or different subspaces. Consider two states $|\psi_{i_1, j_1}^\lambda\rangle \in W_{i_1}^\lambda$ and $|\psi_{i_2, j_2}^\lambda\rangle \in W_{i_2}^\lambda$ that live in subspaces corresponding to the same λ , but potentially different such subspaces (that is, $W_{i_1}^\lambda$ and $W_{i_2}^\lambda$ are potentially different), and suppose that we wanted to test whether they in fact belong to the same subspace (that is, if $i_1 = i_2$). The ability to perform the representation *does not* in general allow us to measure i . Intuitively, this is because both these states behave identically under the representation. A Fourier measurement/sampling of these states would give us only λ , or both λ and j , but not i . So how can we test if they are in the same subspace? This is in general not possible from such a measurement. However, Fourier subspace extraction is more powerful than Fourier measurement and gives us this ability. Performing a Fourier subspace extraction on both states gives us $|\phi_{i_1}^\lambda\rangle |\lambda\rangle |j_1\rangle$ for the first state

¹³Technically, we do need some constraints on the group. We need it to have efficient implementations of a quantum Fourier transform and of its irreducible representations. Note however, that this is a very wide class of groups, and includes, at the very least, all Abelian groups, as well as many important non-Abelian groups. Moreover, *every* fixed-size group is technically efficient (whether Abelian or not), so this condition is important only for some families of groups whose sizes grow exponentially.

and $|\phi_{i_2}^\lambda\rangle|\lambda\rangle|j_2\rangle$ for the second state. Now we can ignore and discard the last register—the one that indicates which state we had within each subspace—and perform a swap test only on the first register, that is between the archetype states that characterize the subspaces. This turns out to be a crucial tool in the security proof of our quantum lightning construction.

The Special Case of Abelian Groups. Abelian groups have the special property that all of the (irreducible) invariant subspaces are one-dimensional. Since the “quantum state” extracted by the Fourier subspace extraction in this case is one-dimensional, it is actually just a complex phase. We can see that the corresponding isometry simplifies to $|\psi_i^\lambda\rangle \mapsto |\phi_i^\lambda\rangle|\lambda\rangle$, where we have absorbed the phase into $|\phi_i^\lambda\rangle$. This is computationally equivalent to the isometry $|\psi_i^\lambda\rangle \mapsto |\psi_i^\lambda\rangle|\lambda\rangle$ (by copying λ and uncomputing), which we can see is just the course Fourier measurement for the representation—that is, a projective measurement onto the subspaces W^λ . We therefore get the following simplified duality for Abelian groups as a special case, a duality between the efficiency of implementing the representation and that of performing a Fourier measurement,¹⁴ a projective measurement on the subspaces spanned by its invariant states.¹⁵

Corollary 1.3 (Duality for Abelian Groups, *informal*). *Let G be an Abelian group and let $\mathcal{F} : G \rightarrow U(\mathcal{H})$ be a representation of G . Then the following are equivalent:*

- \mathcal{F} has an efficient implementation, i.e. $|g\rangle \otimes |\psi\rangle \mapsto |g\rangle \otimes \mathcal{F}(g)|\psi\rangle$.
- \mathcal{F} has an efficient Fourier measurement, i.e. $|\psi_i^\lambda\rangle \mapsto |\psi_i^\lambda\rangle|\lambda\rangle$.

1.3 Related Work

Quantum Money, Lightning, etc. There have been several attempts at constructing public-key quantum money [Aar09, FGH⁺12, AC12, Zha21, KSS22, AGKZ20, KLS22, LMZ23, Zha24]. Unfortunately, a number of them later turned out to be broken [LAF⁺09, CPDDF⁺19, Rob21, LMZ23]. In order to gain confidence in constructions, it is therefore important to give security proofs under computational assumptions that have received significant scrutiny from the cryptographic community. Here, the best we currently have are:

- Quantum money from hidden subspaces [AC12], which was proved secure assuming quantum-resistant *indistinguishability obfuscation* (iO) in [Zha21]. Unfortunately, while candidates for quantum-resistant iO are known, their status is still very much open. This scheme also only achieves quantum money, but not quantum lightning.
- Quantum money from random walks [FGH⁺12, LMZ23], which was shown to be secure under strong quantum “knowledge” assumptions. Such assumptions are not “falsifiable”, and there is some doubt about the plausibility of such assumptions [Zha24].
- Quantum money from Abelian group actions [Zha24], which is proven secure under an assumption *plus* in an idealized model of group actions as a black box.

We provide a scheme whose quantum lightning security we prove in the *plain model* (i.e. without making idealized model assumptions) from a plausible and falsifiable computational assumption. We hope that our work motivates further study of the cryptographic uses of non-Abelian group actions, and in particular, of the hardness of preactions.

¹⁴Note that because representations of Abelian groups have only one-dimensional representations, there is no distinction between the course/weak and the fine/strong versions of Fourier measurement/sampling. Thus, we refer to it as simply Fourier measurement.

¹⁵The invariant subspaces are one-dimensional, and are thus individual quantum states.

Comparison to the Duality of Aaronson, Atia, and Susskind. [AAS20] show that there is a duality between, on the one hand, swapping between two orthogonal states, and on the other hand, measuring the positive and negative superpositions of the two states. As a representation, this “swapping” operation, together with the identity, is a representation of \mathbb{Z}_2 . The invariant subspaces of this representation are the positive and negative superpositions, with eigenvalues $+1$ and -1 . Our duality theorem precisely yields the duality theorem from [AAS20] as a special case when applied to \mathbb{Z}_2 , and recovers the same circuits, showing that our results are a proper generalization.

Theorem 1.2 extends far beyond \mathbb{Z}_2 , and even beyond Abelian groups, to many of the non-Abelian groups that are important for cryptography. We expect our duality theorem to be applicable to many more settings in quantum cryptography and complexity theory. Our applications to building quantum money, lightning, and fire are just a few demonstrations of the usefulness of our theorem and techniques, and demonstrate the usefulness of considering this quantum duality in its full non-Abelian generalization.

2 Preliminaries

2.1 Quantum preliminaries

A register R is a named finite-dimensional Hilbert space. When two registers appear next to each other, as in AB , this refers to the tensor product space of A and B . We write $\text{tr}(\cdot)$ to denote the trace, and $\text{tr}_B(\cdot)$ to denote the partial trace over a register B . We denote by $\|X\|_1 = \text{tr}(|X|)$ the trace norm, where $|X| = \sqrt{XX^\dagger}$. For a vector space V , we write $\text{GL}(V)$ to denote the general linear group from V to itself, i.e. invertible square matrices. For two matrices in $\text{GL}(V)$, we define the Hilbert-Schmidt inner product as follows.

Definition 2.1 (Hilbert-Schmidt inner product). *Let $A, B \in \text{GL}(R)$, then we define the Hilbert-Schmidt inner product between A and B to be*

$$\langle A, B \rangle = \frac{1}{\dim(R)} \text{tr} [AB^\dagger].$$

This implies a norm in the natural way: $\|A\| = \sqrt{\langle A, A \rangle}$.

2.2 Representation Theory

Definition 2.2 (Representation). *Let G be a finite group. Then a function $\mathcal{F} : G \mapsto \text{GL}(R)$ is a representation of G if the following holds for all group elements $g, h \in G$:*

$$\mathcal{F}(g)\mathcal{F}(h) = \mathcal{F}(gh).$$

The vector space R is called a representation space of G . We note that representations need not be defined over Hilbert spaces (they can be defined over any vector space), but we will only ever consider representations that output unitaries in Hilbert spaces. We use the notation $\dim(\mathcal{F})$ to denote the dimension of the representation space R .

We will also need a notion of a function being “almost” a representation. The following is the definition of ϵ -approximate representation (in Hilbert-Schmidt norm), taken from [GH16].

Definition 2.3 (ϵ -approximate representation [GH16]). Let G be a group, and $\mathcal{F} : G \mapsto U(\mathbb{R})$ be a function taking group elements to unitaries over \mathbb{R} . \mathcal{F} is a ϵ -approximate representation if the following holds:

$$\mathbb{E}_{g,h \in G} \left[\operatorname{Re} \left\langle \mathcal{F}(g)^\dagger \mathcal{F}(h), \mathcal{F}(g^{-1}h)^\dagger \right\rangle \right] \geq 1 - \epsilon.$$

Here Re is the real component.

We use the following additional definition of an ϵ -close representation, which is the notation of being close to an exact representation of a group, up to an isometry V .

Definition 2.4 (ϵ -close representation). Let G be a group and $\mathcal{F} : G \mapsto U(\mathbb{R})$ be a function taking group elements to unitaries over \mathbb{R} . We say that \mathcal{F} is ϵ -close to a representation of G if there exists a representation of G , $\mathcal{G} : G \mapsto U(\mathbb{R}')$ and an isometry $V : \mathbb{R} \mapsto \mathbb{R}'$ such that

$$\mathbb{E}_{g \in G} \left\| \mathcal{F}(g) - V^\dagger \mathcal{G}(g) V \right\|^2 \leq \epsilon.$$

We will also need some definitions and facts from character theory. A reference for these can be found in, e.g. [S⁺77].

Definition 2.5 (Irreducible representation). A representation $\rho : G \mapsto \operatorname{GL}(\mathbb{R})$ is an irreducible representation of G if for all subspaces $W \subseteq \mathbb{R}$, $\rho(g)W \not\subseteq W$. We sometimes refer to \mathbb{R} as the irreducible representation of G . Irreducible representations are often called “irreps”.

Definition 2.6 (Dual of a group). The dual of a group G , denoted \widehat{G} , is the set of all irreducible representations of G , up to equivalence by a unitary transformation. For an Abelian group, \widehat{G} will itself have a group structure, but this is not generally the case for non-Abelian groups.

Lemma 2.7 (Size of the dual). The size of the dual, \widehat{G} , of a group is equal to the number of conjugacy classes of G . In particular, for a finite group G , \widehat{G} is also finite.

Definition 2.8 (Character). Let $\mathcal{F} : G \mapsto \operatorname{GL}(\mathbb{R})$ be a representation of G . We define the character of \mathcal{F} to be

$$\chi_{\mathcal{F}}(g) = \operatorname{tr}[\mathcal{F}(g)].$$

Definition 2.9 (Inner product of characters). Let $\chi_{\mathcal{F}}$ and $\chi_{\mathcal{G}}$ be two characters, then we define their inner product to be

$$\langle \chi_{\mathcal{F}} | \chi_{\mathcal{G}} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathcal{F}}(g) \chi_{\mathcal{G}}^\dagger(g).$$

Lemma 2.10 (Irreps are norm 1). For every irreducible representation of a group G , the following holds

$$\langle \chi_{\rho} | \chi_{\rho} \rangle = 1.$$

Lemma 2.11 (Decomposition into irreps). Let \mathcal{F} be a representation of a group G with representation space \mathbb{R} , and let d_{ρ} be $\langle \chi_{\mathcal{F}}, \chi_{\rho} \rangle$. Then the following holds:

$$\mathbb{R} \simeq \bigoplus_{\rho} W_{\rho}^{\oplus d_{\rho}}.$$

Where W_ϱ is the irreducible representation space of ϱ . Furthermore, the decomposition into $W^{\oplus d_\varrho}$ is unique, the decomposition into further subspaces depends on the choice of basis. Furthermore in the basis of $\bigoplus_\varrho W_\varrho^{\oplus d_\varrho}$, $\mathcal{F}(g)$ looks like:

$$\sum_\varrho \Pi_{W_\varrho} \varrho(g) \Pi_{W_\varrho}.$$

Here Π_{W_ϱ} is the projector onto W_ϱ .

Definition 2.12 (Manifestations and multiplicity of an irreducible representation). *The manifestations¹⁶ of an irreducible representation ϱ within a representation \mathcal{F} , are the subspaces W_ϱ that are irreducible representation spaces of ϱ . The number of of manifestations of ϱ in \mathcal{F} , is its multiplicity, which we denote as $n_\varrho^\mathcal{F}$ (or n_ϱ when \mathcal{F} is clear from context). The direct sum of all the manifestations of ϱ is the course Fourier subspace, also known as the isotypic component of ϱ .*

Definition 2.13 (Right/left regular representation). *The left regular representation of a group G is the following function.*

$$\mathcal{L}(h) = \sum_{g \in G} |hg\rangle\langle g|.$$

The right regular representation of a group G is the following function.

$$\mathcal{R}(h) = \sum_{g \in G} |gh^{-1}\rangle\langle g|.$$

Lemma 2.14. *For all groups G and all irreducible representations of G , the following holds*

$$\langle \chi_\mathcal{L} | \chi_\varrho \rangle = \dim(\varrho).$$

and similarly for the right regular representation.

Using this fact, together with the fact that the character of the right (or left) regular representation is equal to $|G|$ at the identity, and 0 elsewhere, we have:

Lemma 2.15. *Let G be a finite group, then the following holds.*

$$\sum_{\varrho \in \hat{G}} \dim(\varrho)^2 = |G|.$$

Definition 2.16 (Plancherel measure). *The Plancherel measure is a probability distribution over irreducible representations of a group G . The Plancherel measure of an irreducible representation ϱ is given by*

$$\mu(\varrho) = \frac{\dim(\varrho)^2}{|G|}.$$

We can see that this corresponds to selecting an irreducible representation according to its “weight” in the sum of [Lemma 2.15](#). A concept we will be interested in is the maximum Plancherel measure of any irreducible representation of the group. For example, for the symmetric group, upper and lower bounds are given by the following lemma.

¹⁶It is common to refer to the different irreducible representation subspaces W_i^ϱ on which the representation \mathcal{F} acts as irrep ϱ as “copies” of the irreducible representation. We prefer the word “manifestations” to avoid confusion later with the notion of copies of a state due to cloning.

Lemma 2.17 (Plancherel measure of the symmetric group [VK85]). *The following inequalities hold for constants $c_0 = 0.2313$ and $c_1 = 2.5651$*

$$e^{-\frac{c_1}{2}\sqrt{n}}\sqrt{n!} \leq \max_{\lambda \in \text{irrep}(S_n)} \dim(\varrho_\lambda) \leq e^{-\frac{c_0}{2}\sqrt{n}}\sqrt{n!}.$$

Thus, the maximum Plancherel measure of an irreducible representation of the symmetric group, S_n , is $e^{-c_0\sqrt{n}}$, which is negligible in n .

Lemma 2.18 (Schur orthogonality relations [Iss05]). *Let $\varrho, \sigma \in \widehat{G}$ be irreducible representations of G . Then we have that:*

$$\sum_{g \in G} \varrho(g)_{ij}^* \sigma(g)_{kl} = \frac{|G|}{\dim(\varrho)} \delta_{\varrho\sigma} \delta_{ik} \delta_{jl}.$$

2.2.1 Quantum Fourier Transform

Now we define the quantum Fourier transform in general.

Definition 2.19 (Quantum Fourier transform). *Let d_ϱ be the dimension of ϱ for every irreducible representation of a group G . The quantum Fourier transform over a general group G is the following unitary transformation*

$$\text{QFT}_G = \sum_{g \in G} \sum_{\substack{\varrho \in \widehat{G}, \\ i, j \in [d_\varrho]}} \sqrt{\frac{d_\varrho}{|G|}} \varrho(g)_{j,i} |\varrho, i, j\rangle \langle g|.$$

Its inverse is

$$\text{QFT}_G^\dagger = \sum_{g \in G} \sum_{\substack{\varrho \in \widehat{G}, \\ i, j \in [d_\varrho]}} \sqrt{\frac{d_\varrho}{|G|}} \varrho(g^{-1})_{i,j} |g\rangle \langle \varrho, i, j|.$$

We will often refer to either one as the quantum Fourier transform over G , and it will be clear from context which one we mean.

We note that for Abelian groups, every irreducible representation is dimension 1, so the sum over i, j goes away, and we recover the usual Abelian quantum Fourier transform.

Definition 2.20 (Fourier basis states). *For a group G , let $\{|\mathcal{L}_{ij}^\varrho\rangle\}_{\varrho \in \widehat{G}, i, j \in [d_\varrho]}$, where $|\mathcal{L}_{ij}^\varrho\rangle := \sqrt{\frac{d_\varrho}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{i,j} |g\rangle$, be the basis recovered by applying QFT_G^\dagger to $\{|\varrho, i, j\rangle\}_{\varrho \in \widehat{G}, i, j \in [d_\varrho]}$. We call this the (left-regular) Fourier basis of G .*

2.3 Fourier Measurements

2.3.1 Coarse Fourier Measurement

Definition 2.21 (Coarse Fourier measurement). *The coarse Fourier measurement¹⁷ is the measurement of the subspaces corresponding to the irreducible representations, but not the basis of the subspaces. Formally, for a group G and representation R , the coarse Fourier measurement is given by the POVM*

$$\left\{ \Pi_{\mathbb{W}_\varrho^{\oplus d_\varrho}} \right\}_{\varrho \in \text{irrep}(G)}.$$

Here the decomposition into unique subspaces $\mathbb{W}_\varrho^{\oplus d_\varrho}$ is given by [Lemma 2.11](#).

Performing a measurement using the coarse Fourier measurement to produce a random irreducible representation label is known in the literature as *weak Fourier sampling*.

2.3.2 Fine Fourier Measurement

Definition 2.22 (Fine Fourier measurement). *Let G be a finite group and \mathcal{F} be a representation of that group. Let W_ϱ be an irreducible representation of G , let $d_\varrho = \langle \chi_{\mathcal{F}} | \chi_\varrho \rangle$, and $\{|\psi_{i,j}^\varrho\rangle\}_{i \in [\dim(\varrho)], j \in [d_\varrho]}$ be a basis for the subspace $W_\varrho^{\oplus d_\varrho}$. Then the fine Fourier measurement¹⁷ is given by the POVM*

$$\{|\psi_{i,j}^\varrho\rangle\langle\psi_{i,j}^\varrho|\}_{\varrho, i, j}.$$

Performing a measurement using the fine Fourier measurement (for any choice of basis) is known in the literature as *strong Fourier sampling*.

2.3.3 Fourier Subspace Extraction

For our purposes, we require a stronger notion than Fourier *measurement*. We introduce a stronger notion called *Fourier subspace extraction*. Unlike Fourier measurement which measures and outputs a classical value for each irreducible representation space W_i^ϱ , Fourier subspace extraction extracts a coherent quantum state out of each W_i^ϱ , maintaining the original superposition within W_i^ϱ but expressing it in the standard basis.

Definition 2.23 (Fourier subspace extraction). *Let G be a finite group and \mathcal{F} be a representation of that group. A Fourier subspace extraction is a coarse projective measurement $\{\Pi_\varrho\}_{\varrho \in \widehat{G}}$ —where each Π_ϱ projects onto $W_\varrho^{\oplus d_\varrho} := \bigoplus_{i \in [n_\varrho]} W_i^\varrho$, the union of the manifestations of ϱ —and a subspace extraction within each subspace W_i^ϱ . Specifically, let each W_i^ϱ have basis $\{|\psi_{ij}^\varrho\rangle\}_{j \in [\dim(\varrho)]}$. Then a Fourier subspace extraction implements a unitary*

$$\mathcal{M} : |\psi_{i,j}^\varrho\rangle |0\rangle \mapsto |\phi_i^\varrho\rangle |\varrho, j\rangle,$$

for some orthonormal set of “archetype” states $\{|\phi_i^\varrho\rangle\}_{\varrho \in \widehat{G}, i \in [n_\varrho]}$.¹⁸

2.4 Group Actions

A group action is a representation of a group that appears often in the field of cryptography. Formally, it is define as follows

Definition 2.24 (Group action). *A group action consists of a family of groups $G = (G_\lambda)_\lambda$, a family of sets $\mathcal{X} = (\mathcal{X}_\lambda)_\lambda$, and a binary operation $* : G_\lambda \times \mathcal{X}_\lambda \mapsto \mathcal{X}_\lambda$ satisfying the following properties*

- **Identity:** *Let $\text{id} \in G$ be the identity element, then $0 * x = x$ for all $x \in \mathcal{X}_\lambda$.*
- **Representation:** *For all $g, h \in G_\lambda$ and $x \in \mathcal{X}_\lambda$, $gh * x = g * (h * x)$.*

We sometimes require the following additional properties.

- **Efficiently computable:** *There is a quantum polynomial-time algorithm that on input 1^λ outputs a description of G_λ and an element $x_\lambda \in \mathcal{X}_\lambda$. The binary operation $*$ is also computable by a quantum polynomial-time algorithm.*

¹⁷ The literature often refers to coarse and fine Fourier measurements as *weak* and *strong* Fourier sampling, respectively. We prefer to use the coarse and fine terminology, capturing how coarse- or fine-grained the decomposition of the space, but we will use the terms interchangeably.

¹⁸Note that the form of the archetype states does not matter. The only requirement is that they are orthonormal so that \mathcal{M} is an isometry. That is, $\langle \psi_{ij}^\varrho | \psi_{k\ell}^\varrho \rangle = \delta_{\varrho\sigma} \delta_{ik} \delta_{j\ell}$.

- **Efficiently recognizable:** There is a quantum polynomial-time algorithm such that for any λ and string y , the algorithm accepts with probability at least $2/3$ if $y \in \mathcal{X}_\lambda$ and rejects with probability at least $2/3$ if $y \notin \mathcal{X}_\lambda$.
- **Transitive:** There is exactly one orbit. That is, for any two elements $x, y \in \mathcal{X}_\lambda$, exists a $g \in G_\lambda$ such that $y = g * x$.
- **Semiregular:** (also called “free”) There are no fixed points. That is, for every $g \in G_\lambda$ and $y \in \mathcal{X}_\lambda$, $g * x = x$ implies that $g = \text{id}$.
- **Regular:** Regular group actions are both transitive and semiregular. That is, for every $y \in \mathcal{X}_\lambda$, there is exactly one $g \in G_\lambda$ such that $y = g * x_\lambda$.

Later in the paper, we will describe additional properties of group actions that will be useful in proving security of cryptographic primitives constructed from group actions.

Definition 2.25 (Orbits of a group action). *The orbit of an element $x \in \mathcal{X}$ is the set of elements accessible from x by acting with G :*

$$\text{Orb}(x) = \{y \mid \exists g \in G \text{ s.t. } y = g * x\}.$$

One important property of group actions is they are representations on the Hilbert space spanned by the elements of \mathcal{X} .

Definition 2.26 (Group Action Representation). *A group action of G defines a representation of G by the following unitary:*

$$F(h) |g * x\rangle = |hg * x\rangle.$$

Note that this representation is a direct sum of left-regular representations on the different orbits of the group action.

2.5 Quantum Money and Quantum Lightning

Now we define public-key quantum money and quantum lightning. Both primitives have the similar syntax, with differences in how their key generation works.

Definition 2.27 (Public-key quantum money [Aar09]). *A public-key quantum money scheme is a triple of efficient quantum algorithms $\mathcal{S} = (\text{KeyGen}, \text{Mint}, \text{Ver})$ where*

- **KeyGen** takes as input the security parameter 1^λ and outputs a private/public key pair (sk, pk) ,
- **Mint**(sk) outputs a pair $(s, |\$^s\rangle)$ where s is a string representing a serial number and $|\$^s\rangle$ is a quantum state representing a banknote,¹⁹ and
- **Ver** takes as input the public key pk , a serial number s , and an alleged banknote σ , and either accepts or rejects.

A public-key quantum money scheme \mathcal{S} satisfies correctness if for all λ ,

$$\Pr \left[\text{Ver}(\text{pk}, s, |\$^s\rangle) \text{ accepts} : \begin{array}{l} (\text{sk}, \text{pk}) \leftarrow \text{KeyGen}(1^\lambda) \\ (s, |\$^s\rangle) \leftarrow \text{Mint}(\text{sk}) \end{array} \right] \geq 1 - \text{negl}(\lambda).$$

¹⁹We will refer to these states interchangeably as either quantum money states or banknotes.

Definition 2.28 (Quantum money security). *A public-key quantum money scheme \mathcal{S} satisfies ϵ -quantum-money security if for all efficient adversaries A , the success probability of A in the counterfeit security game (Algorithm 1) is at most $\epsilon(\lambda)$.*

Algorithm 1 (Public-key Quantum Money Counterfeit Security Game).

1. Generate $(\text{sk}, \text{pk}) \leftarrow \text{KeyGen}(1^\lambda)$, $(s, |\$^s\rangle) \leftarrow \text{Mint}(\text{sk})$ and send $(\text{pk}, s, |\$^s\rangle)$ to the adversary.
2. Adversary returns two registers AB in some potentially entangled state σ_{AB} .
3. Run $\text{Ver}(\text{pk}, s, \sigma_{\text{A}})$ and $\text{Ver}(\text{pk}, s, \sigma_{\text{B}})$. If either check rejects, then reject, otherwise accept.

In place of full public-key quantum money schemes, we will often make use of quantum money *mini-schemes*, simpler objects that can be upgraded to public-key quantum money schemes using digital signatures [AC12]. Because of this effective equivalence, when it is clear from context, we will also often refer to quantum money mini-schemes as public-key quantum money.

Definition 2.29 (Quantum money mini-scheme [AC12]). *A quantum money scheme is a pair of efficient quantum algorithms $\mathcal{S} = (\text{Mint}, \text{Ver})$ where*

- $\text{Mint}(1^\lambda)$ outputs a pair $(s, |\$^s\rangle)$ where s is a string representing a serial number and $|\$^s\rangle$ is the banknote, and
- Ver takes as input a serial number s and an alleged banknote σ , and either accepts or rejects.

The security is similar to that of full public-key quantum money setting:

Algorithm 2 (Quantum Money Mini-Scheme Counterfeit Security Game).

1. Run $(s, |\$^s\rangle) \leftarrow \text{Mint}(1^\lambda)$ and send $(s, |\$^s\rangle)$ to the adversary.
2. Adversary returns two registers AB in some potentially entangled state σ_{AB} .
3. Run $\text{Ver}(s, \sigma_{\text{A}})$ and $\text{Ver}(s, \sigma_{\text{B}})$. If either check rejects, then reject, otherwise accept.

Definition 2.30 (Quantum money mini-scheme security). *A quantum money mini-scheme scheme \mathcal{S} satisfies ϵ -quantum-money security if for all efficient adversaries A , the success probability of A in the counterfeit security game (Algorithm 2) is at most $\epsilon(\lambda)$.*

Quantum lightning is a stronger security guarantee on quantum money in which *not even the mint* can produce two banknotes for the same serial number [Zha21].

Algorithm 3 (Quantum Lightning Counterfeit Security Game).

1. On input 1^λ , adversary returns a serial number s and two registers AB in some potentially entangled state σ_{AB} .
2. Run $\text{Ver}(s, \sigma_A)$ and $\text{Ver}(s, \sigma_B)$. If either check rejects, then reject, otherwise accept.

Definition 2.31 (Quantum lightning security [Zha21]). *A quantum money mini-scheme scheme \mathcal{S} satisfies ϵ -quantum-lightning security if for all efficient adversaries A , the success probability of A in the counterfeit security game (Algorithm 3) is at most $\epsilon(\lambda)$.*

In each of the definitions, when ϵ is a negligible function in λ , we say the scheme satisfies “strong” security.

3 Duality Theorem

In this section we present our main theorem, a computational duality between implementing a group representation and implmenting a Fourier subspace extraction. We first present the exact case in Section 3.1. Then, in Section 3.2, we show how to generalize it to the case of approximate representations and Fourier subspace extractions.

3.1 Exact Case

Theorem 3.1. *Let G be a finite group with an efficient quantum Fourier transform. Let $\mathcal{F} : G \rightarrow U(\mathcal{H})$ be a unitary representation of G , which decomposes into irreducible representations $\{(\varrho, V_i^\varrho)\}_{\varrho \in \widehat{G}, i \in [n_\varrho]}$. Then the following are equivalent:*

1. *There exists a quantum circuit, $C_{\mathcal{F}}$, of size $s_{\mathcal{F}}$, that implements the representation \mathcal{F} . That is, it implements the unitary*

$$|g\rangle \otimes |\psi\rangle \mapsto |g\rangle \otimes \mathcal{F}(g) |\psi\rangle$$

for all $g \in G$ and all $|\psi\rangle \in \mathcal{H}$.

2. *There exists a quantum circuit, $C_{\mathcal{M}}$, of size $s_{\mathcal{M}}$, and²⁰ a quantum circuit, $C_{\mathcal{R}}$, of size $s_{\mathcal{R}}$, where*

- *$C_{\mathcal{M}}$ implements a Fourier subspace extraction, \mathcal{M} , on the Fourier subspaces $\{V_i^\varrho\}_{\varrho \in \widehat{G}, i \in [n_\varrho]}$. That is, $C_{\mathcal{M}}$ implements a coarse projective measurement $\{\Pi_\varrho\}_{\varrho \in \widehat{G}}$ —where each Π_ϱ projects onto $V^\varrho := \bigoplus_{i \in [n_\varrho]} V_i^\varrho$, the union of the manifestations²¹ of ϱ —and a subspace*

²⁰For groups that have an efficiently implementable representation theory—that is, they have efficient implementations of both the quantum Fourier transform as well as for each irrep—the condition of having this second circuit is already satisfied, and can be dropped so as to have a direct relationship between $C_{\mathcal{F}}$ and $C_{\mathcal{M}}$. We include this condition in order to capture a larger class of groups, including groups that have use in our applications, as well as to get a more fine-grained relationship between the complexities.

²¹We refer to the different invariant subspaces V_i^ϱ on which the representation \mathcal{F} acts as irrep ϱ as “manifestations” of ϱ . We do not use the word “copies” to avoid confusion later with the notion of copies of a state due to cloning.

extraction within each subspace V_i^ϱ . Specifically, let each V_i^ϱ have basis $\{|\psi_{ij}^\varrho\rangle\}_{j \in [\dim(\varrho)]}$. Then $C_{\mathcal{M}}$ implements

$$\mathcal{M} : |\psi_{i,j}^\varrho\rangle |0\rangle \mapsto |\phi_i^\varrho\rangle |\varrho, j\rangle ,$$

for some orthonormal set of ‘‘archetype’’ states $\{|\phi_i^\varrho\rangle\}_{\varrho \in \widehat{G}, i \in [n_\varrho]}$.²²

- $C_{\mathcal{R}}$ implements the irreducible representations $\{\varrho\}_{\varrho \in \widehat{G}}$ given an arbitrary catalytic state $|\chi^\varrho\rangle \in V^\varrho$. That is $C_{\mathcal{R}}$ implements²³

$$|\chi^\varrho\rangle |g\rangle \otimes |\psi\rangle \mapsto |\chi^\varrho\rangle |g\rangle \otimes \varrho(g) |\psi\rangle$$

Going from [Item 1](#) to [Item 2](#), we have that $s_{\mathcal{M}}$ and $s_{\mathcal{R}}$ are both $O(s_{\mathcal{F}} + s_{\text{QFT}})$, where s_{QFT} is the circuit complexity of implementing the quantum Fourier transform of G . In the other direction, we have that $s_{\mathcal{F}} = O(s_{\mathcal{M}} + s_{\mathcal{R}})$.

Remark 3.2. In the special case in which the group is Abelian, all the irreducible representations are 1-dimensional, so [Item 2](#) above simplifies to a full projective measurement in the Fourier basis of the representation (the basis of states that are fixed by the representation). Moreover, the quantum Fourier transform for Abelian groups can always be implemented efficiently. Thus we get as a special case that for Abelian groups, the representation is directly dual to a Fourier measurement.

Remark 3.3. As an even more special case, the duality theorem of [\[AAS20\]](#) is the case in which $G \cong \mathbb{Z}_2$.

Proof of [Theorem 3.1](#).

1 \Rightarrow **2**: Suppose that [Item 1](#) is true. That is, we have a circuit of size s that implements the representation \mathcal{F} . Let $\varrho : G \rightarrow U(\mathcal{H})$ be an irrep of G of dimension d_ϱ and multiplicity n_ϱ in \mathcal{F} , and let $V_1^\varrho, \dots, V_{n_\varrho}^\varrho$ be the manifestations of the irrep ϱ in \mathcal{F} . For each subspace V_i^ϱ , take $\{|\psi_{ij}^\varrho\rangle\}_{j \in [\dim(\varrho)]}$ to be a basis for the subspace such that the corresponding irrep unitary $\varrho(g)$ sends $|\psi_{ij}^\varrho\rangle$ to $\sum_{k \in [\dim(\varrho)]} \varrho(g)_{kj} |\psi_{ik}^\varrho\rangle$.²⁴

Suppose we have a basis state $|\psi_{ij}^\varrho\rangle$ on which we want to perform the Fourier subspace extraction to produce $|\phi_i^\varrho\rangle |\varrho, j\rangle$ (for some set of ‘‘archetype’’ states $|\phi_i^\varrho\rangle$).²⁵ We begin by preparing the uniform superposition over the group $\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle$ in an ancilla register and then, controlled on that register, apply the promised circuit $C_{\mathcal{F}}$ for implementing \mathcal{F} to our state.

$$\begin{aligned} & \frac{1}{\sqrt{|G|}} \sum_{g \in G} \mathcal{F}(g) |\psi_{ij}^\varrho\rangle \otimes |g\rangle \\ &= \frac{1}{\sqrt{|G|}} \sum_{g \in G} \sum_k \varrho(g)_{kj} |\psi_{ik}^\varrho\rangle \otimes |g\rangle \\ &= \sum_k |\psi_{ik}^\varrho\rangle \otimes \frac{1}{\sqrt{|G|}} \sum_{g \in G} \varrho(g)_{kj} |g\rangle \end{aligned}$$

²²Note that the form of the archetype states does not matter. The only requirement is that they are orthonormal so that \mathcal{M} is an isometry.

²³Note that the when going from a representation to a Fourier subspace extraction, we end up with an implementation of $C_{\mathcal{R}}$ that uses the catalytic states, but in the other direction, it is sufficient to have a circuit $C_{\mathcal{R}}$ that implements the the irrep controlled on its irrep label, that is, $|\varrho\rangle |g\rangle \otimes |\psi\rangle \mapsto |\varrho\rangle |g\rangle \otimes \varrho(g) |\psi\rangle$.

²⁴Technically, any basis of V_i^ϱ works fine, and we just need to unitarily transform the irrep unitary $\varrho(g)$ accordingly in our minds. However, it is convenient to consider a similar basis for all the subspaces V_i^ϱ corresponding to ϱ , so that we can write $\varrho(g)$ in terms of its matrix elements $\varrho(g)_{ij}$ in the same way across all of them.

²⁵We consider only basis states $|\psi_{ij}^\varrho\rangle$ without loss of generality because the general case follows from linearity.

Inverting the group element in the last register gives

$$\begin{aligned}
&\rightarrow \sum_k |\psi_{ik}^\varrho\rangle \otimes \frac{1}{\sqrt{|G|}} \sum_{g \in G} \varrho(g)_{kj} |g^{-1}\rangle \\
&= \sum_k |\psi_{ik}^\varrho\rangle \otimes \frac{1}{\sqrt{|G|}} \sum_{g \in G} \varrho(g^{-1})_{kj} |g\rangle \\
&= \frac{1}{\sqrt{d_\varrho}} \sum_k |\psi_{ik}^\varrho\rangle \otimes |\mathcal{L}_{kj}^\varrho\rangle,
\end{aligned}$$

where $|\mathcal{L}_{kj}^\varrho\rangle$ is the j th basis vector of the k th manifestation of the irrep ϱ in the *left regular* representation of G . If we now perform a quantum Fourier transform on the second register, we get

$$\frac{1}{\sqrt{d_\varrho}} \sum_k |\psi_{ik}^\varrho\rangle \otimes |\varrho, k, j\rangle.$$

Reordering and regrouping the registers gives us

$$\left(\frac{1}{\sqrt{d_\varrho}} \sum_k |\psi_{ik}^\varrho\rangle \otimes |k\rangle \right) |\varrho, j\rangle = |\phi_i^\varrho\rangle |\varrho, j\rangle$$

We can now measure the register containing ϱ to get the label of the irrep containing our state. Note that within subspace V_i^ϱ , this is a subspace extraction that extracts out $|j\rangle$ —the state in the standard basis corresponding to whichever state inside V_i^ϱ we started with²⁶—and leaves behind the archetype state $|\phi_i^\varrho\rangle := \frac{1}{\sqrt{d_\varrho}} \sum_k |\psi_{ik}^\varrho\rangle \otimes |k\rangle$. Interestingly, observe that in this case, the reduced state on the first register of the archetype state for subspace V_i^ϱ is the fully mixed state on V_i^ϱ . This is not necessarily the case, however, for a general Fourier subspace extraction, which may have any form of archetype state (as long as they form an orthonormal basis for the Fourier subspace extraction to be an isometry).

Note that now that we have a circuit $C_{\mathcal{M}}$ implementing the Fourier subspace extraction, we can easily implement $C_{\mathcal{R}}$ as well. Say we are given as input the state $|\chi^\varrho\rangle |g\rangle |\psi\rangle$, consisting of a catalytic state, $|\chi^\varrho\rangle \in V^\varrho$, indicating the irrep $\varrho \in \widehat{G}$ to compute, a group element $g \in G$, and a state $|\psi\rangle$ on which to compute $\varrho(g)$. Write $|\psi\rangle = \sum_{j \in [\dim(\varrho)]} \alpha_j |j\rangle$ and $|\chi^\varrho\rangle = \sum_{i \in [n_\varrho], j' \in [\dim(\varrho)]} \beta_{ij'} |\psi_{ij'}^\varrho\rangle$. We start by performing the Fourier subspace extraction on $|\chi^\varrho\rangle$, by running $C_{\mathcal{M}}$, to get $\sum_{i \in [n_\varrho], j' \in [\dim(\varrho)]} \beta_{ij'} |\phi_i^\varrho\rangle |\varrho, j'\rangle$. Rearranging the registers, we have, altogether,

$$\sum_{i \in [n_\varrho], j, j' \in [\dim(\varrho)]} \alpha_j \beta_{ij'} |j'\rangle |g\rangle |\phi_i^\varrho\rangle |\varrho\rangle |j\rangle.$$

²⁶Note that the state that is extracted in the last register does not depend on which basis we chose for V_i^ϱ before. In fact, our choice was only a mathematical choice and did not actually affect the computation in any way. What determined the basis we got at the output was really our choice of the vectors $|\mathcal{L}_{kj}^\varrho\rangle$ for the left regular representation, and these are determined simply by which quantum Fourier transform we chose to implement. Interestingly, with a Fourier subspace extraction, since *all* the information about the original state within subspace V_i^ϱ is extracted into a single register in the standard basis, we do not have to decide on a basis ahead of time! We can convert a Fourier subspace extraction in one basis to one in another basis *after the fact* by applying a unitary to the resulting extracted register.

We now uncompute the Fourier subspace extraction on the last three registers by running $C_{\mathcal{M}}^\dagger$. (Note that these are not the same registers that we extracted. We are reversing the Fourier subspace extraction in order to *inject* the state $|\psi\rangle$ into the Fourier subspace V^ϱ where it had never been before!) This gives

$$\sum_{i \in [n_\varrho], j, j' \in [\dim(\varrho)]} \alpha_j \beta_{ij'} |j'\rangle |g\rangle |\psi_{ij}^\varrho\rangle .$$

Now we can run $C_{\mathcal{F}}$ on the last two registers to compute the full representation \mathcal{F} , giving

$$\begin{aligned} & \sum_{i \in [n_\varrho], j, j' \in [\dim(\varrho)]} \alpha_j \beta_{ij'} |j'\rangle |g\rangle \mathcal{F}(g) |\psi_{ij}^\varrho\rangle \\ &= \sum_{i \in [n_\varrho], j, j', k \in [\dim(\varrho)]} \alpha_j \beta_{ij'} \varrho(g)_{kj} |j'\rangle |g\rangle |\psi_{ik}^\varrho\rangle , \end{aligned}$$

where we use the fact that \mathcal{F} acts as ϱ on each of the V_i^ϱ 's. If we now perform another Fourier subspace extraction on the last register, we get

$$\sum_{i \in [n_\varrho], j, j', k \in [\dim(\varrho)]} \alpha_j \beta_{ij'} \varrho(g)_{kj} |j'\rangle |g\rangle |\phi_i^\varrho\rangle |\varrho\rangle |k\rangle ,$$

which we rearrange as

$$\sum_{i \in [n_\varrho], j, j', k \in [\dim(\varrho)]} \alpha_j \beta_{ij'} \varrho(g)_{kj} |\phi_i^\varrho\rangle |\varrho\rangle |j'\rangle |k\rangle |g\rangle ,$$

and we finally uncompute the Fourier subspace extraction on the first three registers, giving

$$\begin{aligned} & \sum_{i \in [n_\varrho], j, j', k \in [\dim(\varrho)]} \alpha_j \beta_{ij'} \varrho(g)_{kj} |\psi_{ij}^\varrho\rangle |k\rangle |g\rangle \\ &= \sum_{i \in [n_\varrho], j' \in [\dim(\varrho)]} \beta_{ij'} |\psi_{ij}^\varrho\rangle |g\rangle \sum_{j, k \in [\dim(\varrho)]} \alpha_j \varrho(g)_{kj} |k\rangle \\ &= \sum_{i \in [n_\varrho], j' \in [\dim(\varrho)]} \beta_{ij'} |\psi_{ij}^\varrho\rangle |g\rangle \sum_{j \in [\dim(\varrho)]} \alpha_j \varrho(g) |j\rangle \\ &= |\chi^\varrho\rangle \otimes |g\rangle \otimes \varrho(g) |\psi\rangle , \end{aligned}$$

This process has therefore sent $|\chi^\varrho\rangle \otimes |g\rangle \otimes |\psi\rangle$ to $|\chi^\varrho\rangle \otimes |g\rangle \otimes \varrho(g) |\psi\rangle$, performing the irrep ϱ on our input state $|\psi\rangle$, as desired.

2 \Rightarrow 1: Suppose that [Item 2](#) is true. Then we have an circuit $C_{\mathcal{M}}$ implementing the Fourier subspace extraction \mathcal{M} , which performs both a projective measurement $\{\Pi_\varrho\}_{\varrho \in \widehat{G}}$, where Π_ϱ projects onto subspace V^ϱ , the (possibly empty) union of some set of subspaces $V_1^\varrho, \dots, V_{n_\varrho}^\varrho$ —where each V_i^ϱ has dimension $\dim(\varrho)$ and is spanned by some basis $\{|\psi_{ij}^\varrho\rangle\}_{j \in [\dim(\varrho)]}$ —and a subspace extraction on each subspace V_i^ϱ :

$$\mathcal{M} : |\psi_{ij}^\varrho\rangle \mapsto |\phi_i^\varrho\rangle |\varrho\rangle |j\rangle$$

for some orthonormal set of archetype states $|\phi_i^\varrho\rangle$.

We would like to perform \mathcal{F} , the representation defined by each $\varrho \in \widehat{G}$ having manifestations $V_1^\varrho, \dots, V_{n_\varrho}^\varrho$. We receive as input a state of the form $|g\rangle |\psi\rangle$, with the first register containing a group element $g \in G$ for which we would like to implement its representation $\mathcal{F}(g)$, and the second register containing a quantum state $|\psi\rangle$ on which we would like to perform the representation.

Write $|\psi\rangle$ in the basis of the $|\psi_{ij}^\varrho\rangle$'s as $|\psi\rangle = \sum_{\varrho, i, j} \alpha_{ij}^\varrho |\psi_{ij}^\varrho\rangle$. We start by applying the promised circuit $C_{\mathcal{M}}$ for implementing \mathcal{M} on $|\psi\rangle$ to get

$$|g\rangle \otimes \mathcal{M} |\psi\rangle = |g\rangle \otimes \sum_{\varrho \in \widehat{G}, i \in [n_\varrho], j \in [\dim(\varrho)]} \alpha_{ij}^\varrho |\phi_i^\varrho\rangle |\varrho\rangle |j\rangle .$$

Now, we use the promised circuit $C_{\mathcal{R}}$ for implementing the irreps of G , applying it²⁷ to perform $\varrho(g)$ on the last register:

$$\begin{aligned} & |g\rangle \otimes \sum_{\varrho \in \widehat{G}, i \in [n_\varrho], j \in [\dim(\varrho)]} \alpha_{ij}^\varrho |\phi_i^\varrho\rangle |\varrho\rangle \otimes \varrho(g) |j\rangle \\ &= |g\rangle \otimes \sum_{\varrho \in \widehat{G}, i \in [n_\varrho], j \in [\dim(\varrho)]} \alpha_{ij}^\varrho |\phi_i^\varrho\rangle \sum_{k \in [d_\varrho]} |\varrho\rangle \otimes \varrho(g)_{kj} |k\rangle \\ &= |g\rangle \otimes \sum_{\varrho \in \widehat{G}, i \in [n_\varrho], j \in [\dim(\varrho)]} \alpha_{ij}^\varrho \sum_{k \in [d_\varrho]} \varrho(g)_{kj} |\phi_i^\varrho\rangle |\varrho\rangle |k\rangle . \end{aligned}$$

We now use $C_{\mathcal{M}}^\dagger$ to uncompute the Fourier subspace extraction on the last three registers, producing

$$\begin{aligned} & |g\rangle \otimes \sum_{\varrho \in \widehat{G}, i \in [n_\varrho], j \in [\dim(\varrho)]} \alpha_{ij}^\varrho \sum_{k \in [d_\varrho]} \varrho(g)_{kj} |\psi_{ik}^\varrho\rangle \\ &= |g\rangle \otimes \sum_{\varrho \in \widehat{G}, i \in [n_\varrho], j \in [\dim(\varrho)]} \alpha_{ij}^\varrho \mathcal{F}(g) |\psi_{ij}^\varrho\rangle \\ &= |g\rangle \otimes \mathcal{F}(g) \sum_{\varrho \in \widehat{G}, i \in [n_\varrho], j \in [\dim(\varrho)]} \alpha_{ij}^\varrho |\psi_{ij}^\varrho\rangle \\ &= |g\rangle \otimes \mathcal{F}(g) |\psi\rangle . \end{aligned}$$

We can see that this successfully implements the representation $\mathcal{F}(g)$. □

From [Theorem 3.1](#), we get the following interesting corollary, which may be of independent interest. It allows us to implement a representation of a group G by using a circuit for implementing a *different* representation of the same group G , as long as the representation we want shares irrep subspaces with the representation we have (or breaks them down further).

²⁷In [Theorem 3.1](#), we allow controlling either on a register containing a description of the irrep label or on a catalytic state $|\chi^\varrho\rangle \in V^\varrho$. In the first case, we can use it directly because we have a register containing a description of the irrep ϱ . In the latter case, one might worry that we no longer have a state in V^ϱ , since the archetype state $|\phi_i^\varrho\rangle$ is not guaranteed to be in V^ϱ or have any particular form. We can always uncompute the Fourier subspace extraction to get back the original state which *is* in fact in the correct subspace V^ϱ , but of course this gets rid of the extracted state we need to perform the irrep on. One might worry that no-cloning would prevent us from having access to both at the same time. Fortunately, this is not the case: the no-cloning principle is not an issue here. We can swap out the register containing $|j\rangle$ and swap in a new register containing an arbitrary state $|\tau\rangle$ —really, any state works—and uncompute the Fourier subspace extraction to inject $|\tau\rangle$ into the subspace V^ϱ . This would allow us to apply $C_{\mathcal{R}}$, after which we can reverse this process, recover and discard $|\tau\rangle$, and proceed as before.

Corollary 3.4. *Given the ability to efficiently implement a representation $\mathcal{F} : G \mapsto U(\mathcal{H})$ that breaks \mathcal{H} into some set of irrep subspaces $\{V_i^\varrho\}_{\varrho \in \widehat{G}, i \in [n_\varrho]}$, we can implement any other representation of the same group that acts on the same irrep subspaces (or further subdivisions of those subspaces), as long as we can implement the irreps of the new representation.*

Main Idea. This results from a double application of [Theorem 3.1](#). Once in the forward direction on the first representation to get a Fourier subspace extraction on the subspaces, and then once in the backwards direction to get an implementation of the second representation. \square

3.2 Approximate Case

Here we present an approximate duality theorem, in that the conditions of [Item 1](#) and [Item 2](#) have to hold approximately. This demonstrates its robustness. In order to prove an approximate version of the duality theorem, we will need the following theorem from [\[GH16\]](#) about approximate representations.

Theorem 3.5 (Gowers-Hatami [\[GH16\]](#)). *Let G be a finite group, $\epsilon \geq 0$ and $\mathcal{F} : G \mapsto U(\mathbb{R})$ be an ϵ -approximate representation of G . Then there exists a register \mathbb{R}' of dimension $d' = (1+O(\epsilon)) \dim(\mathbb{R})$ and an isometry $V : \mathbb{R} \mapsto \mathbb{R}'$ and an exact representation $\mathcal{G} : G \mapsto U(\mathbb{R}')$ such that*

$$\mathbb{E}_{x \in G} \left\| \mathcal{F}(x) - V^\dagger \mathcal{G}(x) V \right\|^2 \leq 2\epsilon.$$

Where the norm $\|\cdot\|$ is implied by the dimension-normalized Hilbert-Schmidt inner product $\langle A, B \rangle = \text{tr}[AB^\dagger] / \dim(\mathbb{R})$.

Remark 3.6. *While the matter of approximate representations has been extensively studied in mathematics and quantum computer science, the idea of an approximate measurement into irreducible representations has not been studied as much. In particular, the idea of weak (or strong) Fourier sampling is typically used in algorithms for solving problems in groups. For these kinds of problems, there is a well defined measurement that one can try to approximate. However in our case, as works from representation theory note [\[GH16, KK82\]](#), there may be vector spaces that admit approximate representations, but for which no exact representation exists. This raises the question of what a measurement into an invariant subspace should look like. [\[GH16\]](#) proposes a lemma pertaining to “approximately invariant subspaces”, but it uses a notion of Fourier transform that is different from the quantum Fourier transform that we often consider. Here we propose a notion of approximate measurement onto an invariant subspace inspired by the result of [\[GH16\]](#), and use it in our duality result.*

Consider the following approximate versions of [Theorem 3.1](#).

Theorem 3.7 (Approximate duality, forward direction). *Let G be a finite group with a Fourier transform that can be implemented with a circuit of size s_{QFT} . Let $\mathcal{F} : G \mapsto U(\mathbb{R})$ be an ϵ -approximate representation of G with a circuit implementation of size $s_{\mathcal{F}}$, with \mathbb{R} being an n -qubit register. Then there exists a register \mathbb{R}' with $n + 1$ qubits, an exact group representation $\mathcal{G} : G \mapsto U(\mathbb{R}')$ and an isometry V mapping \mathbb{R} to \mathbb{R}' such that for the Fourier decomposition $\mathbb{R}' = \bigoplus_{\varrho \in \widehat{G}, i \in [n_\varrho]} V_i^\varrho$ and basis $\{|\psi_{i,j}^\varrho\rangle\}_{\varrho \in \widehat{G}, i \in [n_\varrho], j \in [\dim(\varrho)]}$ of \mathcal{G} as in [Item 2](#) from [Theorem 3.1](#), there is a circuit C of size $O(s_{\mathcal{F}} + s_{\text{QFT}})$, and a set of archetype states $\{|\phi_i^\varrho\rangle\}_{\varrho \in \widehat{G}, i \in [n_\varrho]}$, such that*

$$\frac{1}{\dim(\mathbb{R}')} \sum_{\substack{\varrho \in \widehat{G}, i \in [n_\varrho] \\ j \in [\dim(\varrho)]}} \text{Re} \langle \phi_i^\varrho | \otimes \langle \varrho, j | V C V^\dagger | \psi_{i,j}^\varrho \rangle \geq 1 - \epsilon.$$

That is, we get an ϵ -approximate Fourier subspace extraction.

Proof. We let C be the same circuit as in the exact case, first preparing the state $\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle$, and applying the representation $\mathcal{F}(g)$ controlled on that register to the state. We also let the archetype states be $|\phi_i^\varrho\rangle = \frac{1}{\sqrt{d_\varrho}} \sum_{k \in [d_\varrho]} |\psi_{i,k}^\varrho\rangle \otimes |k\rangle$ as in the exact case, where $d_\varrho := \dim(\varrho)$. We can compute the quantity from the theorem statement as:

$$\begin{aligned}
& \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \langle \phi_i^\varrho | \langle \varrho, j | V C V^\dagger | \psi_{i,j}^\varrho \rangle \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \langle \phi_i^\varrho | \langle \varrho, j | (V \otimes \operatorname{id})(\operatorname{id} \otimes \text{QFT}) \cdot \sum_{g \in G} \mathcal{F}(g) \otimes |g^{-1}\rangle \langle g^{-1}| \cdot (\operatorname{id} \otimes \text{QFT}^\dagger)(V \otimes \operatorname{id})^\dagger | \psi_{i,j}^\varrho \rangle | 0 \rangle \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \frac{1}{\sqrt{d_\varrho}} \sum_k \langle \psi_{i,k}^\varrho | \langle \mathcal{L}_{k,j}^\varrho | (V \otimes \operatorname{id}) \cdot \sum_{g \in G} \mathcal{F}(g) \otimes |g^{-1}\rangle \langle g^{-1}| \cdot (\operatorname{id} \otimes \text{QFT}^\dagger)(V \otimes \operatorname{id})^\dagger | \psi_{i,j}^\varrho \rangle | 0 \rangle \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \frac{1}{\sqrt{d_\varrho}} \sum_k \langle \psi_{i,k}^\varrho | \langle \mathcal{L}_{k,j}^\varrho | (V \otimes \operatorname{id}) \cdot \sum_{g \in G} \mathcal{F}(g) \otimes |g^{-1}\rangle \langle g^{-1}| \cdot (V \otimes \operatorname{id})^\dagger | \psi_{i,j}^\varrho \rangle \frac{1}{\sqrt{|G|}} \sum_{g' \in G} |g'\rangle \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \frac{1}{\sqrt{d_\varrho}} \sum_k \langle \psi_{i,k}^\varrho | \langle \mathcal{L}_{k,j}^\varrho | (V \otimes \operatorname{id}) \cdot \left(\frac{1}{\sqrt{|G|}} \sum_{g \in G} \mathcal{F}(g) \otimes \operatorname{id} \right) \cdot (V \otimes \operatorname{id})^\dagger | \psi_{i,j}^\varrho \rangle | g^{-1} \rangle \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \frac{1}{\sqrt{|G|}} \sum_{g \in G} \frac{1}{\sqrt{d_\varrho}} \sum_k \langle \psi_{i,k}^\varrho | \langle \mathcal{L}_{k,j}^\varrho | (V \otimes \operatorname{id}) \cdot (\mathcal{F}(g) \otimes \operatorname{id}) \cdot (V \otimes \operatorname{id})^\dagger | \psi_{i,j}^\varrho \rangle | g^{-1} \rangle \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \frac{1}{\sqrt{|G|}} \sum_{g \in G} \frac{1}{\sqrt{d_\varrho}} \sum_k \langle \psi_{i,k}^\varrho | \langle \mathcal{L}_{k,j}^\varrho | (V \mathcal{F}(g) V^\dagger \otimes \operatorname{id}) | \psi_{i,j}^\varrho \rangle | g^{-1} \rangle \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \frac{1}{\sqrt{|G|}} \sum_{g \in G} \frac{1}{\sqrt{d_\varrho}} \sum_k \langle \psi_{i,k}^\varrho | V \mathcal{F}(g) V^\dagger | \psi_{i,j}^\varrho \rangle \langle \mathcal{L}_{k,j}^\varrho | g^{-1} \rangle \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \frac{1}{\sqrt{|G|}} \sum_{g \in G} \frac{1}{\sqrt{d_\varrho}} \sum_k \langle \psi_{i,k}^\varrho | V \mathcal{F}(g) V^\dagger | \psi_{i,j}^\varrho \rangle \sqrt{\frac{d_\varrho}{|G|}} \sum_{h \in G} \varrho(h^{-1})_{k,j}^* \langle h | g^{-1} \rangle \quad (1) \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \frac{1}{|G|} \sum_{g \in G} \sum_k \varrho(g)_{k,j}^* \langle \psi_{i,k}^\varrho | V \mathcal{F}(g) V^\dagger | \psi_{i,j}^\varrho \rangle \\
&= \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \frac{1}{|G|} \sum_{g \in G} \langle \psi_{i,j}^\varrho | \mathcal{G}(g)^\dagger V \mathcal{F}(g) V^\dagger | \psi_{i,j}^\varrho \rangle \quad (2) \\
&= \mathbb{E}_{g \in G} \frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \langle \psi_{i,j}^\varrho | \mathcal{G}(g)^\dagger V \mathcal{F}(g) V^\dagger | \psi_{i,j}^\varrho \rangle \\
&= \mathbb{E}_{g \in G} \frac{1}{\dim(\mathbb{R}')} \operatorname{Re} \operatorname{tr} \left[\mathcal{G}(g)^\dagger V \mathcal{F}(g) V^\dagger \right] \quad (3) \\
&= \mathbb{E}_{g \in G} \operatorname{Re} \left\langle \mathcal{F}(g), V^\dagger \mathcal{G}(g) V \right\rangle \quad (4) \\
&= 1 - \frac{1}{2} \mathbb{E}_{g \in G} \left\| \mathcal{F}(g) - V^\dagger \mathcal{G}(g) V \right\|^2 \quad (5) \\
&\geq 1 - \epsilon.
\end{aligned}$$

Here the first line is expanding out the definition of the circuit as a quantum Fourier transform, controlled \mathcal{F} ,²⁸ and then an inverse quantum Fourier transform. (There is also rearrangement of registers, but this is implicit in order to simplify notation.) The second and third lines applies the inverse Fourier transform to the $|0\rangle$ state, which represents the trivial irrep of G , as well as to the $\langle \varrho, k, j |$ (commuting it past the V , which acts only on the first register). The line labeled **1** expands the definition of $|\mathcal{L}_{k,j}^\varrho\rangle$, and line **2** uses the fact that \mathcal{G} exactly performs the representation on the basis of the states $|\psi_{i,j}^\varrho\rangle$. Line **3** uses the fact that the states $|\psi_{i,j}^\varrho\rangle$ form a complete basis for \mathbb{R}' . Line **4** uses the definition of the Hilbert Schmidt inner product, line **5** uses the fact that $\|A - B\| = \sqrt{2 - 2 \operatorname{Re} \langle A, B \rangle}$, and the last line uses the bound from [Theorem 3.5](#). \square

We note that this part of the duality theorem preserves the error between the representation and the measurement.

Remark 3.8. *The forward direction could equivalently be phrased as follows: Let \mathcal{F} be 2ϵ -close to an exact representation \mathcal{G} under isometry V , then there is an implementation of ϵ -approximate Fourier subspace extraction up to V with a circuit whose size of $O(s_{\mathcal{F}} + s_{\text{QFT}} + n)$.*

We can also show the other direction, albeit with (we believe) sub-optimal error scaling.

Theorem 3.9 (Approximate duality, reverse direction). *Let G be a finite group. Let \mathbb{R} and \mathbb{R}' be two registers with an isometry V mapping \mathbb{R} to \mathbb{R}' , and let \mathcal{G} be an exact representation on \mathbb{R}' . Say that we have a circuit $C_{\mathcal{M}}$ of size $s_{\mathcal{M}}$ which implements an ϵ -approximate Fourier subspace extraction in \mathbb{R} , satisfying*

$$\frac{1}{\dim(\mathbb{R}')} \sum_{\varrho, i, j} \operatorname{Re} \langle \phi_i^\varrho | \otimes \langle \varrho, j | (V \otimes \operatorname{id}) \mathcal{M} (V^\dagger \otimes \operatorname{id}) |\psi_{i,j}^\varrho\rangle \otimes |0\rangle \geq 1 - \epsilon.$$

Also assume that we have a circuit of size $s_{\mathcal{R}}$ implementing the irreducible representations of G . Then there exists a circuit of size $O(s_{\mathcal{M}} + s_{\mathcal{R}})$ which implements a map \mathcal{F} of G on \mathbb{R} , that is 2ϵ -close to \mathcal{G} , i.e. one satisfying

$$\mathbb{E}_{g \in G} \left\| V \mathcal{F}(g) V^\dagger - \mathcal{G}(g) \right\|^2 \leq 2\epsilon.$$

Proof. The implementation of \mathcal{F} will be identical to the one from [Theorem 3.1](#). In particular, $\mathcal{F}(g)$ will first apply \mathcal{M} to measure ϱ and extract j , then apply $\varrho(g)$ to the register containing j , and finally it will un-compute \mathcal{M} .

We can proceed by evaluating the average difference between $V \mathcal{F}(g) V^\dagger$ and $\mathcal{G}(g)$ under the Hilbert-Schmidt norm.

$$\begin{aligned} \mathbb{E}_{g \in G} \left\| V \mathcal{F}(g) V^\dagger - \mathcal{G}(g) \right\|^2 &= \mathbb{E}_{g \in G} \langle V \mathcal{F}(g) V^\dagger - \mathcal{G}(g), V \mathcal{F}(g) V^\dagger - \mathcal{G}(g) \rangle \\ &= \mathbb{E}_{g \in G} \frac{1}{\dim(\mathbb{R}')} \operatorname{tr} \left[V \mathcal{F}(g) V^\dagger V \mathcal{F}^\dagger(g) V^\dagger + \mathcal{G}(g) \mathcal{G}(g)^\dagger - V \mathcal{F}^\dagger(g) V^\dagger \mathcal{G}^\dagger(g) - \mathcal{G}(g) V \mathcal{F}(g) V^\dagger \right] \\ &= 2 - \mathbb{E}_{g \in G} \frac{1}{\dim(\mathbb{R}')} \operatorname{tr} \left[V \mathcal{F}(g) V^\dagger \mathcal{G}^\dagger(g) + \mathcal{G}(g) V \mathcal{F}^\dagger(g) V^\dagger \right] \end{aligned} \quad (6)$$

Here we note that the implementation of $\mathcal{F}(g)$ is always unitary, and $V^\dagger V = \operatorname{id}$, so the first two terms are the the identity on \mathbb{R}' . Now we lower bound the second term. We begin by writing it as two times the real component of a trace, and expand the definitions of \mathcal{G} and \mathcal{F} .

²⁸Technically, we control on g^{-1} , but this is just so that we can use the left-regular Fourier transform, rather than the right-regular one. This is not essential, but it slightly simplifies the notation.

$$\begin{aligned}
\mathbb{E}_{g \in G} \operatorname{Re} \frac{2}{\dim(\mathbf{R}')} \operatorname{tr} \left[V \mathcal{F}(g) V^\dagger \mathcal{G}^\dagger(g) \right] &= \mathbb{E}_{g \in G} \frac{2}{\dim(\mathbf{R}')} \operatorname{Re} \sum_{\varrho, i, j} \langle \psi_{i,j}^\varrho | V \mathcal{F}(g) V^\dagger \mathcal{G}(g) | \psi_{i,j}^\varrho \rangle \\
&= \mathbb{E}_{g \in G} \frac{2}{\dim(\mathbf{R}')} \operatorname{Re} \sum_{\varrho, i, j, k} \varrho(g)_{k,j}^\dagger \langle \psi_{i,j}^\varrho | V \mathcal{F}(g) V^\dagger | \psi_{i,k}^\varrho \rangle.
\end{aligned}$$

Now, we expand out the definition of \mathcal{F} . This yields the following state.

$$\begin{aligned}
&\mathbb{E}_{g \in G} \frac{2}{\dim(\mathbf{R}')} \operatorname{Re} \sum_{\varrho, i, j, k} \varrho(g)_{k,j}^\dagger \langle \psi_{i,j}^\varrho | V \mathcal{M}^\dagger \varrho(g) \mathcal{M} V^\dagger | \psi_{i,k}^\varrho \rangle \\
&= \mathbb{E}_{g \in G} \frac{2}{\dim(\mathbf{R}')} \operatorname{Re} \sum_{\substack{\varrho, i, j, k \\ \varrho', a, b}} \varrho(g)_{k,j}^\dagger (\langle \phi_a^{\varrho'} | \otimes \langle \varrho', b |) (V \otimes \operatorname{id}) \mathcal{M} V^\dagger | \psi_{i,k}^\varrho \rangle \langle \psi_{i,j}^\varrho | V \mathcal{M}^\dagger \varrho(g) (V^\dagger \otimes \operatorname{id}) | \phi_a^{\varrho'} \rangle | \varrho', b \rangle \\
&= \mathbb{E}_{g \in G} \frac{2}{\dim(\mathbf{R}')} \operatorname{Re} \sum_{\substack{\varrho, i, j, k \\ \varrho', a, b, c}} \varrho(g)_{k,j}^\dagger \varrho'(g)_{c,b} (\langle \phi_a^{\varrho'} | \otimes \langle \varrho', b |) (V \otimes \operatorname{id}) \mathcal{M} V^\dagger | \psi_{i,k}^\varrho \rangle \langle \psi_{i,j}^\varrho | V \mathcal{M}^\dagger (V^\dagger \otimes \operatorname{id}) | \phi_a^{\varrho'} \rangle | \varrho', c \rangle \\
&= \mathbb{E}_{g \in G} \frac{2}{\dim(\mathbf{R}')} \operatorname{Re} \sum_{\substack{\varrho, i, j, k \\ \varrho', a, b, c}} \varrho(g)_{k,j}^\dagger \varrho'(g)_{c,b} (\langle \phi_a^{\varrho'} | \otimes \langle \varrho', b |) (V \otimes \operatorname{id}) \mathcal{M} V^\dagger | \psi_{i,k}^\varrho \rangle \langle \psi_{i,j}^\varrho | V \mathcal{M}^\dagger (V^\dagger \otimes \operatorname{id}) | \phi_a^{\varrho'} \rangle | \varrho', c \rangle \\
&= \frac{2}{\dim(\mathbf{R}')} \operatorname{Re} \sum_{\substack{\varrho, i, j, k \\ \varrho', a, b, c}} \mathbb{E}_{g \in G} \left[\varrho(g)_{k,j}^\dagger \varrho'(g)_{c,b} \right] (\langle \phi_a^{\varrho'} | \otimes \langle \varrho', b |) (V \otimes \operatorname{id}) \mathcal{M} V^\dagger | \psi_{i,k}^\varrho \rangle \langle \psi_{i,j}^\varrho | V \mathcal{M}^\dagger (V^\dagger \otimes \operatorname{id}) | \phi_a^{\varrho'} \rangle | \varrho', c \rangle \\
&= \frac{2}{\dim(\mathbf{R}')} \operatorname{Re} \sum_{\varrho, a, i, j, k} \frac{1}{\dim(\varrho)} (\langle \phi_a^{\varrho'} | \otimes \langle \varrho, j |) (V \otimes \operatorname{id}) \mathcal{M} V^\dagger | \psi_{i,k}^\varrho \rangle \langle \psi_{i,j}^\varrho | V \mathcal{M}^\dagger (V^\dagger \otimes \operatorname{id}) | \phi_a^{\varrho'} \rangle | \varrho, k \rangle \\
&= \operatorname{Re} \frac{2}{\dim(\mathbf{R}')} \operatorname{tr} \left[\sum_{\varrho, i, j, k} \frac{1}{\dim(\varrho)} \operatorname{id} \otimes | \varrho, j \rangle \langle \varrho, k | (V \otimes \operatorname{id}) \mathcal{M} V^\dagger | \psi_{i,k}^\varrho \rangle \langle \psi_{i,j}^\varrho | V \mathcal{M}^\dagger (V^\dagger \otimes \operatorname{id}) \right] \\
&= \operatorname{Re} \frac{2}{\dim(\mathbf{R}')} \operatorname{tr} \left[\sum_{\varrho, i, j, k} \frac{1}{\dim(\varrho)} \operatorname{id} \otimes | \varrho, j \rangle \langle \varrho, k | \mathcal{M} V^\dagger | \psi_{i,k}^\varrho \rangle \langle \psi_{i,j}^\varrho | V \mathcal{M}^\dagger \right] \\
&= \operatorname{Re} \frac{2}{\dim(\mathbf{R}')} \sum_{\varrho, i, j, k} \frac{1}{\dim(\varrho)} \langle \phi_i^{\varrho'} | \otimes \langle \varrho, k | (V \otimes \operatorname{id}) \mathcal{M} V^\dagger | \psi_{i,k}^\varrho \rangle \langle \psi_{i,j}^\varrho | V \mathcal{M}^\dagger (V^\dagger \otimes \operatorname{id}) | \phi_i^{\varrho'} \rangle \otimes | \varrho, j \rangle \\
&\geq \operatorname{Re} \frac{2}{\dim(\mathbf{R}')} \sum_{\varrho, i} \frac{1}{\dim(\varrho)} \left(\sum_j \langle \phi_i^{\varrho'} | \otimes \langle \varrho, j | (V \otimes \operatorname{id}) \mathcal{M} V^\dagger | \psi_{i,j}^\varrho \rangle \right)^2 \\
&\geq \operatorname{Re} \frac{2}{\dim(\mathbf{R}')} \sum_{\varrho, i, j} \langle \phi_i^{\varrho'} | \otimes \langle \varrho, j | (V \otimes \operatorname{id}) \mathcal{M} V^\dagger | \psi_{i,j}^\varrho \rangle \\
&\geq 2(1 - \epsilon) \\
&\geq 2 - 2\epsilon.
\end{aligned}$$

Here, we insert identity matrices between ϱ and \mathcal{M} , and we use the definition of the inner product. Then, we use the Schur orthogonality relations to cancel the terms where $\varrho \neq \varrho'$ or $(k, j) \neq (c, b)$. Then we use the definition of the trace, and the cyclic property. Finally, since $V \otimes \operatorname{id}$ commutes with $\operatorname{id} \otimes | \varrho, j \rangle \langle \varrho, k |$, we can move it to the other side using the cyclic property. Then we use the fact

that $|\phi_i^g\rangle\langle\phi_i^g| \otimes |\varrho, j\rangle\langle\varrho, k| \preceq \text{id} \otimes |\varrho, j\rangle\langle\varrho, k|$, together with the cyclic property of the trace. Finally, we apply Cauchy Schwarz twice on the sum over j and k , and the assumption about the performance of \mathcal{M} on an average state from $V^\dagger \mathcal{R}'$.

Plugging this back into [Equation \(6\)](#), we get the following upper bound on the average distance between \mathcal{G} and \mathcal{F} :

$$\begin{aligned} \mathbb{E}_{g \in G} \left\| V\mathcal{F}(g)V^\dagger - \mathcal{G}(g) \right\|^2 &\leq 2 - \mathbb{E}_{g \in G} \frac{1}{\dim(\mathcal{R})} \text{tr} \left[V\mathcal{F}(g)V^\dagger \mathcal{G}^\dagger(g) + \mathcal{G}(g)V\mathcal{F}^\dagger(g)V^\dagger \right] \\ &\leq 2 - (2 - 2\epsilon) \\ &\leq 2\epsilon, \end{aligned}$$

as desired. \square

Unlike in [Theorem 3.1](#), we do not show how to recover an efficient approximate implementation of the irreducible representations of G , but rather we assume that they are efficiently implementable. We note that while in the forward direction ([Theorem 3.7](#)), our duality theorem preserves the inner product error from the approximate representation, we are not able to prove a perfectly tight approximate duality because the reverse direction ([Theorem 3.9](#)) yields a different notion of approximate representation, i.e. being close (up to an isometry) to a real representation. Applying the definition of ϵ -approximate representation directly would not yield the same ϵ as we started with in the reverse direction. Note that if we had defined the forward direction in the same way, using the result of [\[GH16\]](#), we would get a perfect duality, but the notion of approximate representation from [Definition 2.3](#) is more widely used. We leave it as an open question whether an ϵ -approximate representation can be recovered in the reverse direction.

Comparison with [\[AAS20\]](#). We comment on how our approximate duality ([Theorems 3.7](#) and [3.9](#)) relates to the approximate duality theorem from [\[AAS20, Theorem 2\]](#). Let $|x\rangle$ and $|y\rangle$ be two orthogonal quantum states and U be a unitary such that

$$\begin{aligned} \langle y | U | x \rangle &= a \\ \langle x | U | y \rangle &= b. \end{aligned}$$

Unlike in the general case of [Theorem 3.7](#), in this case, the fact that $|x\rangle$ and $|y\rangle$ are orthogonal implies that there exists a unitary \widehat{U} in the *same* register such that \widehat{U} exactly swaps $|x\rangle$ and $|y\rangle$. As a representation of \mathbb{Z}_2 , we thus have the efficient ϵ -close representation $\mathcal{F} : g \mapsto U^g$ and an exact representation $\mathcal{G} : g \mapsto \widehat{U}^g$. We then have the following:

$$\begin{aligned} \mathbb{E}_{g \in \mathbb{Z}_2} \left\| U^g - \widehat{U}^g \right\|^2 &= \frac{1}{2} \left(\|\text{id} - \text{id}\|^2 + \|U - \widehat{U}\|^2 \right) \\ &= \frac{1}{2} \langle U - \widehat{U}, U - \widehat{U} \rangle \\ &= \frac{1}{4} \text{tr}[(U - \widehat{U})(U - \widehat{U})^\dagger] \\ &= \frac{1}{4} \left(4 - \text{tr}[U\widehat{U}^\dagger] - \text{tr}[\widehat{U}U^\dagger] \right) \\ &= 1 - \frac{1}{4} \text{Re}(2a + 2b) \\ &= 1 - \frac{\text{Re}(a + b)}{2} =: 2\epsilon. \end{aligned}$$

Here we use the fact that $\text{tr}[U\widehat{U}^\dagger] = \langle x|U|y\rangle + \langle y|U|x\rangle = a + b$ since \widehat{U} exactly swaps $|x\rangle$ and $|y\rangle$, and similarly for $\text{tr}[\widehat{U}U^\dagger]$. Let \mathcal{M} be the measurement implied by the forward direction of the approximate duality (the Fourier subspace extraction simplifies to a binary projective measurement for the case of \mathbb{Z}_2). This is an approximate distinguishing measurement between the states $|\psi\rangle = \frac{|x\rangle+|y\rangle}{\sqrt{2}}$ and $|\phi\rangle = \frac{|x\rangle-|y\rangle}{\sqrt{2}}$ (i.e. the states corresponding to the two one-dimensional irreducible representations of \mathcal{G}), and we calculate the bias below. We assume here without loss of generality that the probability of accepting $|\psi\rangle$ is higher than the probability of accepting $|\phi\rangle$, and that $|0\rangle\langle 0|$ corresponds to the accept outcome. If these are not the case, then the roles of $|\psi\rangle$ and $|\phi\rangle$ or 0 and 1 can be swapped.

$$\begin{aligned}
|(\text{id} \otimes \langle 0|) \mathcal{M} |\psi\rangle|^2 - |(\text{id} \otimes \langle 0|) \mathcal{M} |\phi\rangle|^2 &= |(\text{id} \otimes \langle 0|) \mathcal{M} |\psi\rangle|^2 - \left(1 - |(\text{id} \otimes \langle 1|) \mathcal{M} |\phi\rangle|^2\right) \\
&= |(\text{id} \otimes \langle 0|) \mathcal{M} |\psi\rangle|^2 + |(\text{id} \otimes \langle 1|) \mathcal{M} |\phi\rangle|^2 - 1 \\
&\geq \frac{1}{2}(|(\text{id} \otimes \langle 0|) \mathcal{M} |\psi\rangle| + |(\text{id} \otimes \langle 1|) \mathcal{M} |\phi\rangle|)^2 - 1 \\
&\geq 2 \left(\frac{1}{2} (\text{Re}(\text{id} \otimes \langle 0|) \mathcal{M} |\psi\rangle) + \text{Re}(\text{id} \otimes \langle 1|) \mathcal{M} |\phi\rangle \right)^2 - 1 \\
&\geq 2 \left(\frac{1}{2} + \frac{\text{Re}(a+b)}{4} \right)^2 - 1 \\
&= 2 \left(\frac{1}{4} + \frac{\text{Re}(a+b)}{4} + \frac{\text{Re}(a+b)^2}{16} \right) - 1 \\
&= \frac{\text{Re}(a+b)}{2} + \frac{\text{Re}(a+b)^2}{8} - \frac{1}{2}.
\end{aligned}$$

Here we note that the error bound is much weaker than the tight bound proved in [AAS20]. While our approximate duality theorem is tight with respect to the Hilbert-Schmidt inner product, it does not necessarily recover an optimal *distinguishing* measurement. The bound in [AAS20] in fact modifies the circuit to get a tighter bound, and we comment on this more later.

In the other direction, assume that we have a measurement that accepts $|\psi\rangle$ with probability p and $|\phi\rangle$ probability $p - \Delta$. Then we can first construct a measurement that applies the original measurement, copies the result over, and un-computes the measurement. For this measurement, we have the following:

$$\text{Re}(\text{id} \otimes \langle 0|) \mathcal{M} |\psi\rangle = \sqrt{p}.$$

and similarly

$$\text{Re}(\text{id} \otimes \langle 1|) \mathcal{M} |\phi\rangle = \sqrt{1 - (p - \Delta)}.$$

Note that in this case [Theorem 3.9](#) works up to any unitary applied to $|\psi\rangle$ and $|\phi\rangle$, since they are still orthogonal and thus are a basis for some exact representation of \mathbb{Z}_2 . So we can always pick a unitary on the first register such that the archetype states are exactly the residual states of \mathcal{M} after measuring. Then we have the following bound on the condition of [Theorem 3.9](#):

$$\begin{aligned}
1 - \epsilon &= \frac{1}{2} \left(\sqrt{p} + \sqrt{1 - p + \Delta} \right) \\
&\geq \sqrt{\frac{1 + \Delta}{2}}.
\end{aligned}$$

Here we minimize this expression over p by setting $p = \frac{1+\Delta}{2}$. Let U be the unitary we implement when applying [Theorem 3.9](#) and \widehat{U} be the unitary that swaps $|x\rangle$ and $|y\rangle$. Combined with our

calculation before, we have the following

$$\mathbb{E}_{g \in \mathbb{Z}_2} \left\| U^g - \widehat{U}^g \right\|^2 = 1 - \frac{\operatorname{Re}(a+b)}{2} \leq 2 \left(1 - \sqrt{\frac{1+\Delta}{2}} \right).$$

Here a and b are $\langle x|U|y\rangle$ and $\langle y|U|x\rangle$ respectively. Rearranging terms, we have that

$$\frac{\operatorname{Re}(a+b)}{2} \geq 2\sqrt{\frac{1+\Delta}{2}} - 1 \geq \Delta.$$

The reason we are able to get a tighter duality in this case is because we can alter the measurement *before hand* so that the real component becomes the same as the absolute value, where as to do the same in the forward direction requires modifying the unitary in a way that depends on the group element, and thus would need to be written into the implementation of the duality theorem itself.

Thus, we recover a non-tight version of the approximate duality from [AAS20]. As noted before, in order to get a tighter bound, the approximate duality of [AAS20] analyzes a slightly different algorithm, in which instead of controlling the swap on the positive superposition between $|0\rangle$ and $|1\rangle$, the control qubit is initialized as $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$, with an arbitrary phase that depends on a and b . In our case, this corresponds to initializing the control register with a state that differs from the uniform positive superposition on the group (i.e. the trivial irrep). Specifically, each group element would receive a phase that depends on the Hilbert Schmidt inner product between the ϵ -close representation and some exact representation on g (since we have the freedom to alter the isometry and unitary, we can take any exact representation). This does not work naively, in part because it would seem to require computing an exponential number of complex phases (in the size of the binary representation of the group), but we suspect that such a strategy may be possible in order to get a tighter bound. Since this is not necessary for our case, we leave it to future work.

Remark 3.10. *One might wonder what would happen if we proved a similar theorem, but instead starting from the result of [KK82]. Here, the definition of ϵ -approximate is with respect to the operator norm, but there is no need for an isometry in the resulting exact representation. However, the stricter requirements on this approximate representation make it hard to apply to “approximate adversaries” in the way that we would want. In particular, an adversary that breaks some game with inverse polynomial probability might succeed with very high probability in some cases, but 0 in others. This means that the result of [KK82] does not help us transform these adversaries into other useful adversaries.*

4 Quantum Lightning From Non-Abelian Group Actions

We generalize the construction of quantum money / lightning of [Zha24] to general group actions. This allows us to instantiate the construction from a potentially much wider class of group action instantiations. Generalizing to non-Abelian groups, specifically, also allows us to show a security reduction from a concrete computational assumption *in the plain model*.²⁹ (See Section 4.3 for a discussion of the assumption.) Below, we present a quantum money construction from non-Abelian group actions.

²⁹By contrast, [Zha24] is only able to show a security reduction in the black-box setting of generic group actions.

4.1 The Quantum Lightning Construction

Let G be a group with an efficient quantum Fourier transform and a negligible maximum Plancherel measure (that is, each irrep ϱ of G has dimension at most $d_\varrho := \dim(\varrho) \leq \sqrt{|G|} \cdot \text{negl}(\log |G|)$). For example, we can take G to be the dihedral group D_{2n} or the symmetric group S_n . Let $*$: $G \times X \rightarrow X$ be a semiregular group action of G on some set X , and let $x \in X$ be a fixed starting element in the set. We build a our quantum lightning scheme as follows:

Mint: To mint a quantum bank note, the mint begins with a copy of the starting element of the group $x \in X$ in a quantum register B , in tensor product with the uniform superposition of all elements of the group.³⁰

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle_A |x\rangle_B .$$

The mint then applies the group action, controlled on register A , yielding the following quantum state:

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle_A |g * x\rangle_B .$$

The mint inverts the group element in register A to get:

$$\begin{aligned} \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g^{-1}\rangle_A |g * x\rangle_B &= \frac{1}{\sqrt{|G|}} \sum_{g \in G} \sum_{\substack{\varrho \in \hat{G} \\ i, j \in [d_\varrho]}} \sqrt{\frac{d_\varrho}{|G|}} \varrho(g^{-1})_{ij} |\mathcal{L}_{ji}^\varrho\rangle_A |g * x\rangle_B \\ &= \frac{1}{\sqrt{|G|}} \sum_{\substack{\varrho \in \hat{G} \\ i, j \in [d_\varrho]}} |\mathcal{L}_{ji}^\varrho\rangle_A \sqrt{\frac{d_\varrho}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |g * x\rangle_B , \end{aligned}$$

where $|\mathcal{L}_{ab}^\varrho\rangle := \sqrt{\frac{d_\varrho}{|G|}} \sum_{h \in G} \varrho(h^{-1})_{ab} |h\rangle$ is the Fourier basis state of the left-regular representation.

The mint then applies the quantum Fourier transform on A , yielding the following state:

$$\frac{1}{\sqrt{|G|}} \sum_{\substack{\varrho \in \hat{G} \\ i, j \in [d_\varrho]}} |\varrho, j, i\rangle_A \sqrt{\frac{d_\varrho}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |g * x\rangle_B . \quad (7)$$

The mint then measures A in the computational basis to get an irrep label $\varrho \in \hat{G}$, as well as two Fourier indices $i, j \in [d_\varrho]$. The residual state on register B becomes:

$$|\$\varrho_{ij}\rangle := \sqrt{\frac{d_\varrho}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |g * x\rangle .$$

Output ϱ as the serial number, and $|\$\varrho_{ij}\rangle$ as the quantum money state. This completes the description of Mint.

Lemma 4.1. *The set possible money states $\{|\$\varrho_{ij}\rangle\}_{\varrho \in \hat{G}, i, j \in [d_\varrho]}$ is orthonormal. That is $\langle \$\varrho_{ij} | \$\sigma_{kl} \rangle = \delta_{\varrho\sigma} \delta_{ik} \delta_{jl}$.*

³⁰This can be attained by performing the inverse quantum Fourier transform on the trivial irrep label of the group.

Proof. This follows straightforwardly from Schur orthogonality relations ([Lemma 2.18](#)) and the fact that the group action is semiregular (that is, $g * x = h * x$ only if $g = x$). We have:

$$\begin{aligned}
\langle \$_{ij}^\varrho \mid \$_{kl}^\sigma \rangle &= \frac{\sqrt{d_\varrho d_\sigma}}{|G|} \sum_{g,h \in G} \varrho(g^{-1})_{ij}^* \sigma(h^{-1})_{kl} \langle g * x \mid h * x \rangle \\
&= \frac{\sqrt{d_\varrho d_\sigma}}{|G|} \sum_{g \in G} \varrho(g^{-1})_{ij}^* \sigma(g^{-1})_{kl} \\
&= \frac{\sqrt{d_\varrho d_\sigma}}{|G|} \cdot \frac{|G|}{d_\varrho} \delta_{\varrho\sigma} \delta_{ik} \delta_{jl} \\
&= \delta_{\varrho\sigma} \delta_{ik} \delta_{jl}. \quad \square
\end{aligned}$$

Lemma 4.2. *The serial number—that is, the irrep label ϱ —produced by the Minting is sampled according to the Plancherel measure of ϱ in G . That is, for all $\varrho \in \hat{G}$,*

$$\Pr[\varrho = \sigma \mid (\sigma, \$_{ij}^\sigma) \leftarrow \text{Mint}()] = \frac{d_\varrho^2}{|G|}.$$

Proof. We note that can write [Equation \(7\)](#) as:

$$\frac{1}{\sqrt{|G|}} \sum_{\substack{\varrho \in \hat{G} \\ i,j \in [d_\varrho]}} |\varrho, j, i\rangle_{\mathbf{A}} |\$_{ij}^\varrho\rangle_{\mathbf{B}}.$$

where the $|\$_{ij}^\varrho\rangle$'s are orthonormal by [Lemma 4.1](#). We can see directly that the probability of measuring any triplet of (ϱ, j, i) in register \mathbf{A} is exactly $\frac{1}{|G|}$. Furthermore, since for each $\varrho \in \hat{G}$, i and j both run over $[d_\varrho]$, ϱ appears in d_ϱ^2 such triplets. The total probability of the mint outputting serial number ϱ is therefore $\frac{d_\varrho^2}{|G|}$, which is the Plancherel measure of ϱ . \square

Lemma 4.3. *For each $\varrho \in \hat{G}$ and each $i \in [d_\varrho]$, the set $\{|\$_{ij}^\varrho\rangle\}_{j \in [d_\varrho]}$ spans a manifestation, $V_{i,x}^\varrho$, of irrep ϱ in the group action representation $\mathcal{A}(h) = \sum_{g \in G} |hg * x\rangle\langle g * x|$.*

Proof. Applying $\mathcal{A}(h)$ to $|\$_{ij}^\varrho\rangle$ gives us:

$$\begin{aligned}
\mathcal{A}(h) |\$_{ij}^\varrho\rangle &= \sqrt{\frac{d_\varrho}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |hg * x\rangle \\
&= \sqrt{\frac{d_\varrho}{|G|}} \sum_{g' = hg \in G} \varrho((g')^{-1}h)_{ij} |g' * x\rangle \\
&= \sqrt{\frac{d_\varrho}{|G|}} \sum_{g \in G} (\varrho(g^{-1}) \varrho(h))_{ij} |g * x\rangle \\
&= \sqrt{\frac{d_\varrho}{|G|}} \sum_{g \in G} \sum_{k \in [d_\varrho]} \varrho(g^{-1})_{ik} \varrho(h)_{kj} |g * x\rangle \\
&= \sum_{k \in [d_\varrho]} \varrho(h)_{kj} \sum_{g \in G} \varrho(g^{-1})_{ik} |g * x\rangle \\
&= \sum_{k \in [d_\varrho]} \varrho(h)_{kj} |\$_{ik}^\varrho\rangle.
\end{aligned}$$

We can see that $\mathcal{A}(h)$ acts exactly as the irrep ϱ on the space spanned by the money states $\{|\mathcal{S}_{ij}^\varrho\rangle\}_{j \in [d_\varrho]}$. They must therefore span the same manifestation $V_{i,x}^\varrho$ of irrep ϱ in $\mathcal{A}(h)$. \square

Corollary 4.4. *For all $\varrho \in \hat{G}$ and $i, j \in [d_\varrho]$, the money state $|\mathcal{S}_{ij}^\varrho\rangle$ is in the subspace $V_x^\varrho = \bigoplus_{i \in [d_\varrho]} V_{i,x}^\varrho$ corresponding to irrep ϱ of the group action representation starting from x . Moreover, if the group action has multiple orbits, then the full isotypic component of ϱ is $V^\varrho = \bigoplus_{y \in \text{Orb}(\mathcal{A})} V_y^\varrho$ where y runs over the set of orbits of the group action, choosing an element from each orbit arbitrarily.*

Ver: To verify, we begin by measuring that the state has support only on the set X . We then repeat essentially the same process as for minting, but starting with the claimed banknote in the second register, rather than $|x\rangle$. Suppose we want to verify a state $|\mathcal{Y}^\varrho\rangle$ with claimed serial number ϱ , we prepare the uniform superposition over group elements, perform the group action on $|\mathcal{Y}^\varrho\rangle$ in superposition, and then measure the control register in the Fourier basis. That is, we perform a course Fourier measurement on $|\mathcal{Y}^\varrho\rangle$ and check if it has the claimed label.

Suppose that $|\mathcal{Y}^\varrho\rangle$ is a valid state for label ϱ . That is $|\mathcal{Y}^\varrho\rangle = \sum_{i,j \in [d_\varrho]} \alpha_{ij} |\mathcal{S}_{ij}^\varrho\rangle$ for some coefficients α_{ij} . This gives the following:

$$\begin{aligned}
\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |\mathcal{Y}^\varrho\rangle &= \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \sum_{i,j \in [d_\varrho]} \alpha_{ij} |\mathcal{S}_{ij}^\varrho\rangle \\
&\xrightarrow{\text{group action}} \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \sum_{i,j,k \in [d_\varrho]} \alpha_{ij} \varrho(g)_{kj} |\mathcal{S}_{ik}^\varrho\rangle \\
&= \sum_{i,j,k \in [d_\varrho]} \alpha_{ij} \frac{1}{\sqrt{|G|}} \sum_{g \in G} \varrho(g)_{kj} |g\rangle |\mathcal{S}_{ik}^\varrho\rangle \\
&\xrightarrow[g]{\text{invert}} \sum_{i,j,k \in [d_\varrho]} \alpha_{ij} \frac{1}{\sqrt{|G|}} \sum_{g \in G} \varrho(g)_{kj} |g^{-1}\rangle |\mathcal{S}_{ik}^\varrho\rangle \\
&= \sum_{i,j,k \in [d_\varrho]} \alpha_{ij} \frac{1}{\sqrt{|G|}} \sum_{g \in G} \varrho(g^{-1})_{kj} |g\rangle |\mathcal{S}_{ik}^\varrho\rangle \\
&= \sum_{i,j,k \in [d_\varrho]} \frac{\alpha_{ij}}{\sqrt{d_\varrho}} |\mathcal{L}_{kj}^\varrho\rangle |\mathcal{S}_{ik}^\varrho\rangle
\end{aligned}$$

Now if we perform a course (left-regular) Fourier basis measurement on the first register (perform a Fourier transform and measure the irrep label) we get the correct serial number ϱ . Now we have two options: we can further perform a fine Fourier basis measurement to get k and j , collapsing the quantum money state to a new state $|\mathcal{S}'^\varrho\rangle = \sum_{i \in [d_\varrho]} \alpha'_{ik} |\mathcal{S}_{ik}^\varrho\rangle$ with different weights on the same set $\{|\mathcal{S}_{ab}^\varrho\rangle\}_{a,b \in [d_\varrho]}$ of basis states, but nevertheless still a valid quantum money state. Or, alternatively, we can refrain from measuring k and j and simply uncompute the whole process, which, in the case that verification passed with certainty, recovers the original state $|\mathcal{Y}^\varrho\rangle$.

Remark 4.5. *Note that states of the form $\sum_{i,j \in [d_\varrho]} \alpha_{ij} |\mathcal{S}_{ij}^\varrho\rangle$ are not the only states that pass verification. If we denote $|\mathcal{S}_{ij}^\varrho * x\rangle := \sqrt{\frac{d_\varrho}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |g * x\rangle$ as the quantum money state produced by beginning with starting set element $x \in X$, then states of the form $|\mathcal{S}_{ij}^\varrho * y\rangle$ for all $y \in X$ and $i, j \in [d_\varrho]$ (and their superpositions) also pass verification. Thus they are also valid quantum money states, despite not being the result of the minting process, and must be considered in the security arguments.*

4.2 Variations on the construction

Here, we describe some possible variations of the above scheme.

Membership check. The set X may be a collection of sparse strings. In this case, a quantum money adversary may try to forge with fake banknotes that have support outside of X . If the group action supports membership testing for X , it is natural to also have the verifier check that a supposed banknote has support on X . Such a check is used in [Zha24] to analyze the security of their construction. For many group actions, however, such a membership check is not efficiently feasible. In the case in which the group still acts compatibly (or approximately so) on elements outside of X which cannot be distinguished from X , then this can be treated as the group action having additional orbits. In Section 4.3.3, we show an example of how to handle such a group action.

Irrep check. It may be useful to insist that the serial number of the banknote corresponds to an irrep with certain properties. Notably, we will consider adding checks on the *dimension* of the irrep, assuming the dimension is efficiently computable. For example, we may insist that banknotes come from irreps of dimension at least 2.

For such irrep checks, in order to ensure correctness, we need to ensure that the mint always produces irreps with the given property. If such irreps are at least inverse-polynomially dense according to the Plancherel measure, we can have the mint keep minting banknotes until it produces one with the given property.

The following lemma shows that the irreps of size at least 2 are dense for all non-Abelian groups. Thus, for any non-Abelian group action, we can insist on valid banknotes having irrep dimension at least 2. We will make this assumption in the security analysis of our scheme in the following subsections.

Lemma 4.6. *For any non-Abelian G , let d be the dimension of a random irrep sampled according to the Plancherel measure. Then $\Pr[d \geq 2] \geq 1/2$.*

Proof. The 1-dimensional irreps are in bijection with the quotient of the commutator subgroup $G/[G, G]$. Since $|[G, G]| \geq 2$ for non-Abelian groups, $|G/[G, G]| \leq |G|/2$. The probability of sampling any given 1-dimensional irrep according to the Plancherel measure is $1/|G|$. Over all $\leq |G|/2$ such irreps, the probability of sampling *any* 1-dimensional irrep is at most $1/2$. This means the probability sampling an irrep of dimension 2 or higher is at least $1/2$. \square

4.3 Security from pre-action secure group actions

In this subsection we give a security proof from cryptographic group actions that are pre-action-secure, which we define here. A *pre-action* on a group action is an operation that on input $x, g * x \in X$ computes $h \circ (g * x) := gh^{-1} * x$ for some $h \in G$.³¹ That is, it prepends a group element h^{-1} on the right of g , as if h^{-1} had acted *before* g had acted. While for Abelian group actions, this is equivalent to the group action itself (up to inverting h), for non-Abelian groups, this is not generically efficient. Note, however that a preaction *is* itself a group action, as it satisfies compatibility—that is, $h_1 \circ (h_2 \circ (g * x)) = (h_1 h_2) \circ (g * x)$. And moreover, the preaction of a preaction is the original group action.

We introduce both a search-type assumption and a decision-type assumption, that constitute different levels of preaction security. The search-type assumption, preaction hardness, requires

³¹We assume here, as before, that the action is semiregular, so that this is well-defined.

that it is computationally hard to perform a preaction. The decision-type assumption, preaction indistinguishability, requires that it is hard to tell when a preaction has been performed. In both cases, the preactions are defined relative to a predetermined and fixed starting element $x \in X$.

Assumption 1 (Preaction Hardness). *Given $g * x$, and h for random $g, h \leftarrow G$, it is hard to output $gh^{-1} * x$. That is, for all QPT adversaries, \mathcal{A} ,*

$$\Pr [z = gh^{-1} * x : x \leftarrow X, g, h \leftarrow G, z \leftarrow \mathcal{A}(x, g * x, h)] \leq \frac{1}{|G|} + \epsilon$$

Algorithm 4 (Preaction Indistinguishability Security Game).

1. Challenger samples $b \in \{0, 1\}$ and two uniformly random group elements $h_1, h_2 \leftarrow G$.
2. Adversary sends a register A to the challenger.
3. If $b = 0$, the challenger applies the action of h_1 to A . Otherwise, the challenger applies both the action of h_1 and the preaction of h_2 to A . Send A back to the adversary.
4. Adversary outputs b' and wins if $b' = b$.

Assumption 2 (Preaction Indistinguishability). *It is hard to distinguish whether a preaction has been performed. Formally, no adversary can win at the preaction indistinguishability security game (Algorithm 4) with advantage greater than ϵ . That is, if we write the action of the challenger in Step 3 as $F_b^{h_1, h_2} : g * x \mapsto h_1 g h_2^{-b} * x$. Then for all QPT adversaries, \mathcal{A} , that make a single query³² to F ,*

$$\left| \Pr [0 \leftarrow \mathcal{A}^{F_0^{h_1, h_2}} : h_1, h_2 \leftarrow G] - \Pr [0 \leftarrow \mathcal{A}^{F_1^{h_1, h_2}} : h_1, h_2 \leftarrow G] \right| \leq \epsilon$$

Note that when $b = 0$, $F_b^{h_1, h_2}$ performs a group action for a random group element h_1 , and when $b = 1$, it performs both a random bi-action—that is, a random group action with h_1 and a random group pre-action with h_2 .

Definition 4.7. We say that a group action of group G_λ on set X_λ with starting element x is ϵ -preaction secure if both Assumption 1 and Assumption 2 hold for the group action against any QPT adversary with advantage ϵ . We say that the group action is preaction secure if it is $\text{negl}(\lambda)$ -preaction secure for any negligible function negl .

Remark 4.8. Classically, distinguishing preactions in one round is information-theoretically impossible. This is because both cases—with or without a preaction—send the element to a uniformly random element in its orbit. Interestingly, as we will see, this is not the case for quantum distinguishers, since they are allowed to query $F_b^{h_1, h_2}$ on superpositions of elements.

³²We could in general define multi-round security game for preaction indistinguishability, in which Steps 2 and 3 are repeated. Preaction security defined this way would be a stronger assumption, and may be useful for other settings. However, we do not formally define the stronger version as we are able to prove security from this weaker assumption, which gives a stronger security guarantee, and is more likely to hold for a larger class of group actions.

Remark 4.9. For Abelian group actions, breaking preaction hardness is trivial, since the preaction is equal to the action. On the other hand, for this same reason, preaction indistinguishability is information-theoretically impossible: since the preaction is equal to the action, both cases—with or without a preaction—end up performing a uniformly random group action.

Because of Remark 4.9, preaction security is a security notion that only makes sense for non-Abelian group actions. Moreover, the security proof for our quantum money scheme makes explicit use of the properties of representations of non-Abelian groups to prove the reduction.

In fact, for quantum adversaries and non-Abelian group actions, preaction indistinguishability is a stronger assumption than preaction hardness:

Theorem 4.10. Let $(G, X, *, x)$ be a semiregular single-orbit group action of a non-Abelian group G acting on set X . Then if the group action satisfies preaction indistinguishability with advantage ϵ , then it also satisfies preaction hardness with advantage $\theta(\epsilon)$.

Proof sketch. We defer the proof of Theorem 4.10 to Section 4.3.2 because it makes use of the quantum money construction of Section 4.3.1. The main idea is that preactions are themselves a representation of the group, with the Fourier indices exchanging roles relative to their roles for the group action. Thus for semiregular single-orbit group actions, the ability to perform preactions allows us to measure in which manifestation of the irrep a state lies via the duality theorem (Theorem 3.7), with distinguishing error ϵ on average for a random group element. We can then distinguish if a preaction has occurred by testing if it has moved us to a different manifestation of the irrep. \square

Therefore, when working with non-Abelian group actions that are semiregular and single-orbit, it suffices for preaction security to consider only the decision-type assumption of the indistinguishability of preactions.

4.3.1 Construction.

Let (G, X, x) be a semiregular single-orbit group action satisfying the requirements of Section 4.1. The quantum money/lightning construction follows the framework of Section 4.1. We show below that if the group action is preaction secure, then the quantum money construction satisfies lightning security.

4.3.2 Security

Let $|\$\!_{ij}^e\rangle \propto \sum_{g \in G} \varrho(g^{-1})_{ij} |g * x\rangle$ be the quantum money states minted by the scheme. We show a tight connection between the ability to perform preaction (i.e. breaking preaction hardness, Assumption 1) and performing a “right representation” on the quantum money state, that is, coherently mapping the quantum money state as $|\$\!_{ij}^e\rangle \mapsto \sum_{k \in [\dim(\varrho)]} \varrho(h^{-1})_{ik} |\$\!_{kj}^e\rangle$ on input $h \leftarrow G$. This right representation treats the span of the same vector across the different manifestations of ϱ as a single invariant irrep subspace. In other words, it is to the standard group action representation what the right-regular representation is to the left-regular representation. We say that an adversary can perform the right representation with advantage ϵ if it can perform a unitary with Hilbert-Schmidt inner product at least $\frac{1}{|G|} + \epsilon$ with the ideal right representation.

Lemma 4.11. Any adversary that performs a preaction $x, g * x, h \mapsto gh^{-1} * x$ with advantage ϵ for a fixed starting element, $x \in X$, and random $g, h \leftarrow G$, can be used to perform a right representation, $|\$\!_{ij}^e\rangle \mapsto \sum_{k \in [\dim(\varrho)]} \varrho(h^{-1})_{ik} |\$\!_{kj}^e\rangle$ with the same advantage ϵ . Similarly any adversary that performs

a right representation with advantage ϵ . can be used to perform a preaction with the same advantage ϵ .

Proof. Consider an adversary that performs preactions with advantage ϵ , and consider what happens in the ideal case, in which the preaction is performed exactly. We start with the money state

$$|\$\!_{ij}^{\varrho}\rangle \propto \sum_{g \in G} \varrho(g^{-1})_{ij} |g * x\rangle .$$

We perform a preaction with h to get

$$\begin{aligned} & \rightarrow \sum_{g \in G} \varrho(g^{-1})_{ij} |gh^{-1} * x\rangle \\ & = \sum_{g \in G} \varrho(h^{-1}g^{-1})_{ij} |g * x\rangle \\ & = \sum_{\substack{g \in G \\ k \in [\dim(\varrho)]}} \varrho(h^{-1})_{ik} \varrho(g^{-1})_{kj} |g * x\rangle \\ & = \sum_{k \in [\dim(\varrho)]} \varrho(h^{-1})_{ik} \sum_{g \in G} \varrho(g^{-1})_{kj} |g * x\rangle \\ & \propto \sum_{k \in [\dim(\varrho)]} \varrho(h^{-1})_{ik} |\$\!_{ij}^{\varrho}\rangle . \end{aligned}$$

Let $U = \sum_{h, \varrho, i, k, j} |h\rangle\langle h| \otimes \varrho(h^{-1})_{i,k} |\$\!_{k,j}^{\varrho}\rangle\langle\$\!_{i,j}^{\varrho}|$ be the unitary that performs the right representation controlled on a group element h . We can therefore rewrite it as $U = \sum_{h, g} |h\rangle\langle h| \otimes |gh^{-1} * x\rangle\langle g * x|$.

Now suppose that the adversary performs preactions with advantage ϵ . That is, it performs some $\tilde{U} = \sum_{h, g} |h\rangle\langle h| \otimes |\psi(h, g * x)\rangle\langle g * x|$ where the probability of success is $\frac{1}{|G|^2} \sum_{h, g} \langle gh^{-1} * x | \psi(h, g * x)\rangle = \frac{1}{|G|} + \epsilon$. Then we have that, by the definition of the Hilbert-Schmidt inner product,

$$\langle U, \tilde{U} \rangle = \frac{1}{|G|^2} \sum_{h, g} \langle gh^{-1} * x | \psi(h, g * x)\rangle = \frac{1}{|G|} + \epsilon .$$

Conversely, consider an adversary that performs the right representation with advantage ϵ . That is, it performs some operator \tilde{U} such that $\langle U, \tilde{U} \rangle = \frac{1}{|G|} + \epsilon$, where $U = \sum_{h, \varrho, i, k, j} |h\rangle\langle h| \otimes \varrho(h^{-1})_{i,k} |\$\!_{k,j}^{\varrho}\rangle\langle\$\!_{i,j}^{\varrho}|$ is the ideal unitary that performs the right representation.

Consider what happens when the ideal unitary is run on $|g * x\rangle$. We start by writing $|g * x\rangle$ in the basis of the quantum money states $\{|\$\!_{ij}^{\varrho}\rangle\}_{\varrho \in \hat{G}, i, j \in [\dim(\varrho)]}$:

$$|g * x\rangle \propto \sum_{\substack{\varrho \in \hat{G} \\ i, j \in [\dim(\varrho)]}} \varrho(g)_{ji} |\$\!_{ij}^{\varrho}\rangle .$$

Now we perform the right representation to get

$$\begin{aligned}
&\rightarrow \sum_{\substack{\rho \in \widehat{G} \\ i, j \in [\dim(\rho)]}} \varrho(g)_{ji} \sum_{i' \in [\dim(\rho)]} \varrho(h^{-1})_{i, i'} |\$\!_{i', j}^\rho\rangle \\
&= \sum_{\substack{\rho \in \widehat{G} \\ i, i', j \in [\dim(\rho)]}} \varrho(g)_{ji} \varrho(h^{-1})_{i, i'} |\$\!_{i', j}^\rho\rangle \\
&= \sum_{\substack{\rho \in \widehat{G} \\ i', j \in [\dim(\rho)]}} \varrho(gh^{-1})_{j, i'} |\$\!_{i', j}^\rho\rangle \\
&\propto |gh^{-1} * x\rangle .
\end{aligned}$$

If instead we run \tilde{U} on $|g * x\rangle$, and measure in the computational basis, we get the correct preaction with probability

$$\begin{aligned}
\frac{1}{|G|^2} \sum_{h, g \in G} \langle h | \otimes \langle gh^{-1} * x | \tilde{U} |h\rangle \otimes |g * x\rangle &= \frac{1}{|G|^2} \sum_{h, g \in G} \langle h | \otimes \langle g * x | U^\dagger \tilde{U} |h\rangle \otimes |g * x\rangle \\
&= \langle \tilde{U}, U \rangle = \frac{1}{|G|} + \epsilon . \quad \square
\end{aligned}$$

Therefore, pre-action hardness of the group action ([Assumption 1](#)) is equivalent to the hardness of performing the right representation on the money states to map one manifestation of an irrep to another manifestation of the same irrep.

Corollary 4.12. *For a group action to be δ -preaction secure, at most a fraction $\frac{1}{|G|} + \delta$ of the Plancherel measure of G can be on irreps of dimension 1.*

Proof sketch. If a fraction at least $\frac{1}{|G|} + \delta$ of the Plancherel measure falls on 1-dimensional irreps, then the right representation can be simulated to advantage δ by a left representation (that is, the original group action representation), and so by [Lemma 4.11](#), would break preaction hardness ([Assumption 1](#)). \square

Remark 4.13. *By [Corollary 4.12](#), we see that for any preaction-secure group action, the event of sampling a multi-dimensional irrep from the Plancherel measure happens with overwhelming probability, strengthening [Lemma 4.6](#), which states that it happens with probability at least $\frac{1}{2}$ for general non-Abelian group actions. We can therefore always assume that the quantum money state sampled by the minting algorithm lies in a multi-dimensional irrep. We will assume therefore for the rest of the section that the quantum money verification rejects such 1-dimensional irreps.*

Corollary 4.14. *An adversary for preaction hardness with advantage ϵ can be used to perform a **right-Fourier** measurement on the quantum money state with that outputs the correct index i of the manifestation of ρ with advantage ϵ . That is, it can be used to measure i for quantum money state $|\$\!_{ij}^\rho\rangle \propto \sum_{g \in G} \varrho(g^{-1})_{ij} |g * x\rangle$.*

Proof sketch. From [Lemma 4.11](#), we can use the preaction hardness adversary to perform a $1 - (\frac{1}{|G|} + \epsilon)$ -close right representation on the quantum money state. By [Theorem 3.7](#), we get a right-Fourier subspace extraction that, for a uniformly random quantum money state $|\$\!_{i, j}^\rho\rangle$, measures the correct i with advantage at least ϵ . \square

We now show the following lemma which completes the proof of [Theorem 4.10](#), showing that preaction indistinguishability implies preaction hardness for non-Abelian group actions.

Lemma 4.15. *An adversary that can perform a **right-Fourier** measurement on the quantum money state with advantage ϵ can be used to break preaction indistinguishability ([Assumption 2](#)) with advantage $\frac{\epsilon}{2}$.*

Proof. Assume at first that we have a perfect such adversary for performing right-Fourier measurements. We start by using it to measure i on a uniformly random quantum money state $|\$_{ij}^g\rangle$, which can be prepared by running the minting algorithm (which by [Lemma 4.2](#) produces ϱ sampled according to the Plancharel measure and i and j sampled uniformly from $[\dim(\varrho)]$). We then apply the challenger given by [Assumption 2](#) to get

$$\begin{aligned}
& \rightarrow \sum_{g \in G} \varrho(g^{-1})_{ij} |h_1 g h_2^{-b} * x\rangle \\
& = \sum_{g \in G} \varrho(h_2^{-b} g^{-1} h_1)_{ij} |g * x\rangle \\
& = \sum_{\substack{g \in G \\ k, \ell \in [\dim(\varrho)]}} \varrho(h_2^{-b})_{ik} \varrho(g^{-1})_{k\ell} \varrho(h_1)_{\ell j} |g * x\rangle \\
& = \sum_{k, \ell \in [\dim(\varrho)]} \varrho(h_2^{-b})_{ik} \varrho(h_1)_{\ell j} \sum_{g \in G} \varrho(g^{-1})_{k\ell} |g * x\rangle \\
& = \sum_{k, \ell \in [\dim(\varrho)]} \varrho(h_2^{-b})_{ik} \varrho(h_1)_{\ell j} |\$_{k\ell}^g\rangle \tag{8}
\end{aligned}$$

Suppose that $b = 1$. Then when averaged over all pairs of group elements, h_1 and h_2 , this gives

$$\begin{aligned}
& \frac{1}{|G|^2} \sum_{h_1, h_2 \in G} \sum_{k, k', \ell, \ell' \in [\dim(\varrho)]} \varrho(h_2)_{ik}^* \varrho(h_1)_{\ell j}^* \varrho(h_2)_{ik'} \varrho(h_1)_{\ell' j} |\$_{k\ell}^g\rangle \langle \$_{k'\ell'}^g| \\
& = \frac{1}{|G|^2} \sum_{k, k', \ell, \ell' \in [\dim(\varrho)]} \sum_{h_1 \in G} \varrho(h_1)_{\ell j}^* \varrho(h_1)_{\ell' j} \sum_{h_2 \in G} \varrho(h_2)_{ik}^* \varrho(h_2)_{ik'} |\$_{k\ell}^g\rangle \langle \$_{k'\ell'}^g| \\
& = \frac{1}{\dim(\varrho)^2} \sum_{k, k', \ell, \ell' \in [\dim(\varrho)]} \delta_{\ell\ell'} \delta_{kk'} |\$_{k\ell}^g\rangle \langle \$_{k'\ell'}^g| \\
& = \frac{1}{\dim(\varrho)^2} \sum_{k, \ell \in [\dim(\varrho)]} |\$_{k\ell}^g\rangle \langle \$_{k\ell}^g|,
\end{aligned}$$

where the second equality follows from the Schur orthogonality relations ([Lemma 2.18](#)). This is the fully mixed state over the isotypic component of ϱ —that is, over the union of all of the manifestations of irrep ϱ .

Now with probability $1 - \frac{1}{\dim(\varrho)} \geq \frac{1}{2}$ (since $\dim(\varrho) \geq 2$), we get that $k \neq i$. That is, with probability at least $\frac{1}{2}$, the quantum money state has moved to a different manifestation of the irrep ϱ , and measuring it again will confirm this.

If instead $b = 0$, then $k = i$ with certainty (as $\varrho((h_2^0))_{ik} = \varrho(\text{id})_{ik} = \delta_{ik} \quad \forall \varrho \in \widehat{G}, h_2 \in G$). So we output $b' = 1$ if $k \neq i$ and 0 otherwise. This gives a distinguishing advantage of at least $\frac{1}{2}$, breaking [Assumption 2](#).

Now suppose that the adversary has advantage ϵ . In the case where there is no preaction, the adversary will always measure the same i , *even* if they perform a Fourier measurement that is noisy, as long as they perform a unitary transformation. This is because when there is no preaction, it will never change the manifestation. In the case when the adversary performs a right Fourier measurement with advantage ϵ , they will have probability at least $\epsilon(1 - \frac{1}{\dim(\varrho)})$ chance of measuring a different manifestation label (ϵ from the measurement inaccuracy, and the other term from the probability that the manifestation actually moved). Thus, they break preaction indistinguishability with probability $\epsilon \left(1 - \frac{1}{\dim(\varrho)}\right) \geq \frac{\epsilon}{2}$. \square

We can now complete the proof of [Theorem 4.10](#) by combining [Lemmas 4.11](#) and [4.15](#) and [Corollary 4.14](#). \square

We now turn to the quantum lightning security of the scheme. We argue that any adversary who has two copies of the quantum money state can use them to break preaction indistinguishability ([Assumption 2](#)). We therefore get a secure quantum lightning scheme from any group action that satisfies the syntactic requirements and is preaction-secure.

Focusing on the archetype states. For the analysis, before we proceed, it will be useful to consider a proxy for the quantum money states. The money states lie in a potentially large subspace, which is harder to analyze, so it is useful to instead focus on the archetype state that appears after performing a Fourier subspace extraction, which is a unique state that characterizes each such subspace.

Suppose we have a quantum money state $|\mathcal{S}_{ij}^{\varrho}\rangle$. We perform a Fourier subspace extraction using [Theorem 3.1](#), and get

$$|\mathcal{S}_{ij}^{\varrho}\rangle \xrightarrow{FSE} |\phi_i^{\varrho}\rangle |\varrho\rangle |j\rangle = \left(\frac{1}{\sqrt{d_{\varrho}}} \sum_k |\mathcal{S}_{ik}^{\varrho}\rangle \otimes |k\rangle \right) |\varrho\rangle |j\rangle$$

Observation 4.16. *We observe that the archetype state $|\phi_i^{\varrho}\rangle$ in the first register is unaffected by applying the group action:*

$$|\mathcal{S}_{ij}^{\varrho}\rangle \xrightarrow{\text{action by } h} \sum_{\ell} \varrho(h)_{\ell j} |\mathcal{S}_{i\ell}^{\varrho}\rangle \xrightarrow{FSE} |\phi_i^{\varrho}\rangle |\varrho\rangle \left(\sum_{\ell} \varrho(h)_{\ell j} |\ell\rangle \right)$$

On the other hand, applying the corresponding preaction performs the (inverted) irrep ϱ onto the set of archetype states $\{|\phi_i^{\varrho}\rangle\}_{i \in [\dim(\varrho)]}$ for the different manifestations of ϱ :

$$|\mathcal{S}_{ij}^{\varrho}\rangle \xrightarrow{\text{preaction by } h} \sum_{\ell} \varrho(h^{-1})_{i\ell} |\mathcal{S}_{\ell j}^{\varrho}\rangle \xrightarrow{FSE} \left(\sum_{\ell} \varrho(h^{-1})_{i\ell} |\phi_{\ell}^{\varrho}\rangle \right) |\varrho\rangle |j\rangle$$

Proposition 4.17. *Suppose that the group action used in the quantum money construction ([Section 4.3.1](#)) is ϵ -preaction secure. Then no QPT adversary can produce a quantum state on two registers such that the probability of measuring both registers in the same irreducible representation subspace ϱ is greater than $2 \dim(\varrho)\epsilon/(1 + \dim(\varrho))$.*

Proof. Assume for the sake of contradiction that an adversary for quantum lightning, \mathcal{A} , can prepare a quantum state on two registers, both of which pass verification. By definition, the

verifier projects onto V^ϱ . Since we have shown that $|\$\!_{i,j}^\varrho\rangle$ is a basis for V^ϱ , the states produced by \mathcal{A} must be supported on states of the form

$$|\$\!_{ij}^\varrho\rangle \otimes |\$\!_{k\ell}^\varrho\rangle \quad \text{where } |\$\!_{ij}^\varrho\rangle = \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |g * x\rangle$$

for some $\varrho \in \widehat{G}$ such that $1 < \dim(\varrho)$. We show that this adversary can be used to break [Assumption 2](#). Let F_b be the challenger given in the assumption, which either applies a random action and random pre-action ($b = 1$), or just applies a random action ($b = 0$).

To demonstrate the idea, we first assume that $i = k$, that is, that the two registers initially lie in the same Fourier subspace of ϱ . We will see later how to handle the more general case. Suppose that we take only one of the two registers and apply F_b . We get (see [Equation \(8\)](#))

$$\sum_{r,s \in [\dim(\varrho)]} \varrho(h_2^{-b})_{ir} \varrho(h_1)_{sj} |\$\!_{rs}^\varrho\rangle = \begin{cases} \sum_{s \in [\dim(\varrho)]} \varrho(h_1)_{sj} |\$\!_{is}^\varrho\rangle & b = 0 \\ \sum_{r,s \in [\dim(\varrho)]} \varrho(h_2^{-1})_{ir} \varrho(h_1)_{sj} |\$\!_{rs}^\varrho\rangle & b = 1 \end{cases}$$

Then if $b = 0$ (i.e. the challenger did not apply a pre-action), the state remains in the same Fourier subspace with certainty, and so a swap test between the archetype states produced by performing a Fourier subspace extraction on both registers will succeed with probability 1, and we output $b' = 0$.

If $b = 1$, then with probability $1 - \frac{1}{\dim(\varrho)} \geq \frac{1}{2}$ (since $\dim(\varrho) \geq 2$), the resulting state is in a different Fourier subspace. In this case, the swap test between the archetype states fails with probability $\frac{1}{2}$, in which case we output $b' = 1$. Thus, in this case, we output 1 with probability at least $\frac{1}{4}$. The overall success probability is therefore $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$, breaking [Assumption 2](#).

However, the initial states need not lie in the same initial Fourier subspace, so instead we give the following algorithm that sandwiches an application of F between two applications of the symmetric subspace projector. Formally, consider the following algorithm.

Algorithm 5. *Adversary for pre-action indistinguishability given a two-register state, both with support on the same irrep ϱ .*

Input: Two quantum registers that are in valid money states for ϱ and a query to the blackbox $F_b^{h_1, h_2}$ given by [Assumption 2](#).

1. Perform **Fourier subspace extraction** on the two halves of the input.
2. Perform a **swap test** between the two registers containing the archetype states produced.
3. Uncompute the **Fourier subspace extraction** on both halves of the state.
4. Query $F_b^{h_1, h_2}$ on the first register.
5. Perform **Fourier subspace extraction** on both halves of the state.
6. Perform a second **swap test** between the two registers containing the archetype states produced.
7. If the results of both the first and second swap tests agree, output $b' = 0$ (“no preaction”).
8. If the results of the two swap tests disagree, output $b' = 1$ (“preaction”).

Case 1: $b = 0$ (there is no preaction). We first claim that in the case where there is no preaction, the algorithm outputs “no preaction” with probability 1. In order to argue this, we analyze the case when the adversary measures the symmetric subspace in the first measurement *after* performing the Fourier subspace extraction, and argue that un-computing the subspace extraction, applying $F_0^{h_1, h_2}$, and then performing Fourier subspace extraction *always* maps us back into the symmetric subspace on the first register. Since $B_0^{h_1, h_2}$ is a unitary, this will imply that it is block diagonal in Π^{sym} and Π^{asym} . In this case, all sums will go from 0 to $\dim(\varrho) - 1$, so we drop the summands.

Recall that the symmetric subspace is equal to the span of $|\psi\rangle^{\otimes 2}$, for $|\psi\rangle = \sum_{i,j} \alpha_i |\phi_i^\varrho\rangle$, so we can write the state after measuring the symmetric subspace as being in the span of:

$$\left(\sum_{i,k} \alpha_i \alpha_k |\phi_i^\varrho\rangle \otimes |\phi_k^\varrho\rangle \right) \otimes \sum_{j,\ell} \beta_{j,\ell} |\varrho, \varrho, j, \ell\rangle .$$

Inverting the Fourier subspace extraction, we get the following state

$$\sum_{i,k,j,\ell} \alpha_i \alpha_k \beta_{j,\ell} |\$_{i,j}^\varrho\rangle \otimes |\$_{k,\ell}^\varrho\rangle .$$

Then, after applying $F_0^{h_1, h_2}$ to the first register of this state, we have the following.

$$\sum_{i,j,k,\ell} \sum_s \alpha_i \alpha_k \beta_{j,\ell} \varrho(h_1)_{s,j} |\$_{i,s}^\varrho\rangle \otimes |\$_{k,\ell}^\varrho\rangle .$$

Then after performing subspace extraction on both registers, we end up with the following state

$$\begin{aligned} & \sum_{i,j,k,\ell} \sum_s \alpha_i \alpha_k \beta_{j,\ell} \varrho(h_1)_{s,j} (|\phi_i^\varrho\rangle \otimes |\phi_k^\varrho\rangle) \otimes |\varrho, \varrho, s, \ell\rangle \\ &= \sum_{i,k} \alpha_i \alpha_k (|\phi_i^\varrho\rangle \otimes |\phi_k^\varrho\rangle) \otimes \sum_{j,s,\ell} \beta_{j,\ell} \varrho(h_1)_{s,j} |\varrho, \varrho, s, \ell\rangle \\ &= \sum_{i,k} \alpha_i \alpha_k (|\phi_i^\varrho\rangle \otimes |\phi_k^\varrho\rangle) \otimes \sum_{s,\ell} \sum_j (\beta_{j,\ell} \varrho(h_1)_{s,j}) |\varrho, \varrho, s, \ell\rangle . \end{aligned}$$

Setting $\beta'_{s,\ell} = \sum_j \beta_{j,\ell} \varrho(h)_{s,j}$, we get that we are still in the symmetric subspace within the first register. Since this applied to any setting of coefficients, the unitary transformation that composes steps 2, 3 and 4 preserves the symmetric and anti-symmetric subspaces. Thus, if the first measurement has either outcome, the second measurement on step 6 will have the same outcome with probability 1, and the adversary will output ‘no preaction’ with probability 1.

Case 2: $b = 1$ (there is a preaction). We perform a similar analysis in the case where there is a pre-action, but now we will need to consider both subspaces. This is because we need to prove that the unitary that the adversary implements in steps 3 through 5 maps *every* vector from the symmetric subspace to something with high overlap with the anti-symmetric subspace, and vice versa. Starting with the symmetric subspace, we have the same starting state after inverting the Fourier subspace extraction.

$$\sum_{i,k,j,\ell} \alpha_i \alpha_k \beta_{j,\ell} |\$_{i,j}^\varrho\rangle \otimes |\$_{k,\ell}^\varrho\rangle .$$

After applying F , we will end up with the following state

$$\sum_{i,k,j,\ell,r,s} \alpha_i \alpha_k \beta_{j,\ell} \varrho(h_2^{-1})_{i,r} \varrho(h_1)_{s,j} |\$_{r,s}^\varrho\rangle \otimes |\$_{k,\ell}^\varrho\rangle .$$

After performing Fourier subspace extraction, we end up with the following state.

$$\begin{aligned} & \sum_{i,k,j,\ell,r,s} \alpha_i \alpha_k \beta_{j,\ell} \varrho(h_2^{-1})_{i,r} \varrho(h_1)_{s,j} (|\phi_r^\varrho\rangle \otimes |\phi_k^\varrho\rangle) |\varrho, \varrho, s, \ell\rangle \\ &= \left(\sum_{r,k} \left(\sum_i \alpha_i \varrho(h_2^{-1})_{i,r} \right) \alpha_k |\phi_r^\varrho\rangle \otimes |\phi_k^\varrho\rangle \right) \otimes \sum_{s,\ell} \left(\sum_j \beta_{j,\ell} \varrho(h_1)_{s,j} \right) |\varrho, \varrho, s, \ell\rangle \\ &= \left(\sum_r \alpha'_r |\phi_r^\varrho\rangle \right) \otimes \left(\sum_k \alpha_k |\phi_k^\varrho\rangle \right) \otimes \sum_{s,\ell} \left(\sum_j \beta_{j,\ell} \varrho(h_1)_{s,j} \right) |\varrho, \varrho, s, \ell\rangle . \end{aligned}$$

Here in the final line we define $\alpha'_r = \sum_i \alpha_i \varrho(h_2^{-1})_{i,r}$. We can then write out the following expression for the inner product of the first two registers with their swap.

$$\begin{aligned} \text{F}_{\text{SWAP}} &= \left(\sum_r (\alpha'_r)^\dagger \langle \phi_r^\varrho | \right) \otimes \left(\sum_k \alpha_k^\dagger \langle \phi_k^\varrho | \right) \text{SWAP} \left(\sum_{r'} \alpha'_{r'} |\phi_{r'}^\varrho\rangle \right) \otimes \left(\sum_{k'} \alpha_{k'} |\phi_{k'}^\varrho\rangle \right) \\ &= \sum_{r,k} \left((\alpha'_r)^\dagger \langle \phi_r^\varrho | \right) \otimes \left(\alpha_k^\dagger \langle \phi_k^\varrho | \right) \text{SWAP} \left(\alpha'_k |\phi_k^\varrho\rangle \right) \otimes \left(\alpha_r |\phi_r^\varrho\rangle \right) \\ &= \sum_{r,k} \left((\alpha'_r)^\dagger \alpha_k^\dagger \alpha_r \alpha'_k \right) . \end{aligned}$$

Now we analyze a single term in the sum. Since α' itself is a sum of more elements, this will make the equations more managable.

$$\begin{aligned} (\alpha'_r)^\dagger \alpha_k^\dagger \alpha_r \alpha'_k &= \sum_{i,i'} \alpha_i^\dagger \varrho(h_2^{-1})_{i,r}^\dagger \alpha_k^\dagger \alpha_r \alpha_i \varrho(h_2)_{i',k}^{-1} \\ &= \alpha_k^\dagger \alpha_r \sum_{i,i'} \alpha_i^\dagger \alpha_{i'} \varrho(h_2^{-1})_{i,r}^\dagger \varrho(h_2^{-1})_{i',k} . \end{aligned}$$

Now, computing an average over group elements and adding back in the sum over r and k , we have the following:

$$\begin{aligned} \sum_{r,k} \alpha_k^\dagger \alpha_r \sum_{i,i'} \alpha_i^\dagger \alpha_{i'} \mathbb{E}_{h_2 \in G} \varrho(h_2^{-1})_{i,r}^\dagger \varrho(h_2^{-1})_{i',k} &= \frac{1}{\dim(\varrho)} \left(\sum_r \alpha_r^\dagger \alpha_r \right) \left(\sum_i \alpha_i^\dagger \alpha_i \right) \\ &= \frac{1}{\dim(\varrho)} . \end{aligned}$$

Here we use the fact that $\langle \$_{ab}^\varrho | \$_{cd}^\varrho \rangle = \frac{\dim(\varrho)}{|G|} \sum_{h \in G} \varrho(h^{-1})_{a,b}^* \varrho(h^{-1})_{c,d} = \delta_{ac} \delta_{bd}$ ([Lemma 2.18](#)) to cancel out the terms for which $r \neq k$ and $i \neq i'$, and then we use the fact that α_i come from a normalized quantum state. To complete the proof, the probability that the swap test accepts on the state is given by

$$\frac{1}{2} (1 + \text{F}_{\text{SWAP}}) = \frac{1}{2} + \frac{1}{2 \dim(\varrho)} .$$

This means that *every* vector in the symmetric state gets mapped to a vector with overlap $1/2 + 1/2 \dim(\varrho)$ with the anti-symmetric state. Thus, if the first swap test returned the symmetric subspace, the second one returns the symmetric subspace with this probability.

Now, we need to analyze the anti-symmetric subspace. Similar to before, we take a basis for the anti-symmetric subspace and analyze what happens. There is a simple basis described by the $\binom{\dim(\varrho)}{2}$ vectors of the form

$$\frac{1}{\sqrt{2}} \left(|\$\!_{i,j}^\varrho\rangle \otimes |\$\!_{k,\ell}^\varrho\rangle - |\$\!_{k,j}^\varrho\rangle \otimes |\$\!_{i,\ell}^\varrho\rangle \right).$$

Going through the same steps, after applying F , now with a pre-action, we have the following state

$$\frac{1}{\sqrt{2}} \sum_{r,s} \left(\varrho(h_2^{-1})_{i,r} \varrho(h_1)_{s,j} |\$\!_{r,s}^\varrho\rangle \otimes |\$\!_{k,\ell}^\varrho\rangle - \varrho(h_2^{-1})_{k,r} \varrho(h_1)_{s,j} |\$\!_{r,s}^\varrho\rangle \otimes |\$\!_{i,\ell}^\varrho\rangle \right).$$

Now we can examine the probability that a state starting from the symmetric subspace is still in the symmetric subspace (and that a state starting from the anti-symmetric subspace is still in the anti-symmetric subspace) after the Fourier subspace extraction and swap test. When we perform Fourier subspace extraction, we have the following state

$$\begin{aligned} |\psi_{i,j,k,\ell}\rangle &= \frac{1}{\sqrt{2}} \sum_{r,s} \left(\varrho(h_2^{-1})_{i,r} \varrho(h_1)_{s,j} |\phi_r^\varrho\rangle \otimes |\phi_k^\varrho\rangle - \varrho(h_2^{-1})_{k,r} \varrho(h_1)_{s,j} |\phi_r^\varrho\rangle \otimes |\phi_i^\varrho\rangle \right) \otimes |\varrho, \varrho, s, \ell\rangle \\ &= \frac{1}{\sqrt{2}} \sum_r \left(\varrho(h_2^{-1})_{i,r} |\phi_r^\varrho\rangle \otimes |\phi_k^\varrho\rangle - \varrho(h_2^{-1})_{k,r} |\phi_r^\varrho\rangle \otimes |\phi_i^\varrho\rangle \right) \otimes \sum_s \varrho(h_1)_{s,j} |\varrho, \varrho, s, \ell\rangle. \end{aligned}$$

Since the operations up until now were unitary, we can write every state in the anti-symmetric subspace as a linear combination of vectors of this form. $\sum_{i,j,k,\ell} \alpha_{i,j,k,\ell} |\psi_{i,j,k,\ell}\rangle$. We need to compute the inner product between this state and the swapped version of this state, which we can compute as

$$\begin{aligned}
& \mathbb{E}_{h_2 \in G} \left[\sum_{\substack{i,j,k,\ell \\ i',j',k',\ell'}} \alpha_{i,j,k,\ell} \alpha_{i',j',k',\ell'}^\dagger \langle \psi_{i,j,k,\ell} | \text{SWAP} | \psi_{i',j',k',\ell'} \rangle \right] \\
&= \mathbb{E}_{h_2 \in G} \left[\frac{1}{2} \sum_{i,k,k',i'} \sum_{r,r'} (\varrho(h_2^{-1})_{i',r}^* \langle \phi_r^\varrho | \otimes \langle \phi_{k'}^\varrho | - \varrho(h_2^{-1})_{k',r}^* \langle \phi_r^\varrho | \otimes \langle \phi_{i'}^\varrho | \right. \\
&\quad \left. (\varrho(h_2^{-1})_{i,r'} \langle \phi_{k'}^\varrho | \otimes | \phi_{r'}^\varrho \rangle - \varrho(h_2^{-1})_{k,r'} \langle \phi_i^\varrho | \otimes | \phi_{r'}^\varrho \rangle) \left(\sum_{j,\ell} \alpha_{i,j,k,\ell} \alpha_{i',j,k',\ell}^\dagger \right) \right] \\
&= \mathbb{E}_{h_2 \in G} \left[\frac{1}{2} \sum_{i,k,k',i'} \sum_{r,r'} \left(\varrho(h_2^{-1})_{i',r}^* \varrho(h_2^{-1})_{i,r'} \langle \phi_r^\varrho | \phi_{k'}^\varrho \rangle \langle \phi_{k'}^\varrho | \phi_{r'}^\varrho \rangle - \varrho(h_2^{-1})_{i',r}^* \varrho(h_2^{-1})_{k,r'} \langle \phi_r^\varrho | \phi_i^\varrho \rangle \langle \phi_{k'}^\varrho | \phi_{r'}^\varrho \rangle \right. \right. \\
&\quad \left. \left. - \varrho(h_2^{-1})_{k',r}^* \varrho(h_2^{-1})_{i,r'} \langle \phi_r^\varrho | \phi_{k'}^\varrho \rangle \langle \phi_{i'}^\varrho | \phi_{r'}^\varrho \rangle + \varrho(h_2^{-1})_{k',r}^* \varrho(h_2^{-1})_{k,r'} \langle \phi_r^\varrho | \phi_i^\varrho \rangle \langle \phi_{i'}^\varrho | \phi_{r'}^\varrho \rangle \right) (\beta_{i,k,k',i'}) \right] \\
&= \mathbb{E}_{h_2 \in G} \left[\frac{1}{2} \sum_{i,k,k',i'} \left(\varrho(h_2^{-1})_{i',k}^* \varrho(h_2^{-1})_{i,k'} - \varrho(h_2^{-1})_{i',i}^* \varrho(h_2^{-1})_{k,k'} \right. \right. \\
&\quad \left. \left. - \varrho(h_2^{-1})_{k',k}^* \varrho(h_2^{-1})_{i,i'} + \varrho(h_2^{-1})_{k',i}^* \varrho(h_2^{-1})_{k,i'} \right) \beta_{i,k,k',i'} \right] \\
&= \sum_{i,k} \frac{1}{2} \beta_{i,k,k,i} \left(\mathbb{E}_{h_2 \in G} \left[|\varrho(h_2^{-1})_{i,k}|^2 + |\varrho(h_2^{-1})_{k,i}|^2 \right] \right).
\end{aligned}$$

In the first equality, the swap only affects the first two registers, so the final two indices must be the same to survive the inner product. In the third equality, we use the fact that $\langle \phi_a^\varrho | \phi_b^\varrho \rangle = \delta_{ab}$. In getting to the final line, we use the fact that the i indices are never equal to the k indices, by the fact that we are in the anti-symmetric group. Combining this with the fact from before that $\sum_{h \in G} \varrho(h^{-1})_{a,b}^* \varrho(h^{-1})_{c,d} = \delta_{c,d} \delta_{b,d}$, we can remove the two negative terms when averaging over the group elements. Using the same fact, we have that for the remaining terms, we have

$$\frac{1}{|G|} \sum_{h_2 \in G} |\varrho(h_2^{-1})_{i,k}|^2 + \frac{1}{|G|} \sum_{h_2 \in G} |\varrho(h_2^{-1})_{k,i}|^2 = \frac{2}{\dim(\varrho)}.$$

Since the $\beta_{i,k,k,i}$ sum to 1 (as they are again the norm of the original vectors), the probability that the swap test succeeds on the second try is exactly

$$\frac{1}{2} + \frac{1}{2} \left(\frac{1}{\dim(\varrho)} \right).$$

Now, we have shown that in the case when there is a pre-action, for *all* states, the probability that the second swap test succeeds is given by

$$\frac{1}{2} + \frac{1}{2 \dim(\varrho)}.$$

Since the adversary accepts whenever the results are different, the adversary outputs “preaction” with probability at least

$$\frac{1}{2} - \frac{1}{2 \dim(\varrho)}.$$

This is also the distinguishing advantage, as we showed that in the case where there is no pre-action, the adversary outputs “no pre-action” with probability 1.

If the adversary starts with a state that is $2 \dim(\varrho)\epsilon/(\dim(\varrho) + 1)$ close to the tensor product space of ϱ , they can first simply measure the irrep label of both states, and conditioned on getting ϱ for both run this test. If the probability they measure ϱ for both is at least the given probability, then their distinguishing advantage will be at least ϵ . \square

4.3.3 Generalizing to Intransitive Group Actions

Previously, we assumed that the group action was *transitive*. That is, it had a single orbit, such that every element of the set X can be reached from a single starting point $x \in X$. In this subsection, we generalize to the case in which the group action is *intransitive*. This means that the space is divided up into multiple orbits, with each orbit operating as a new manifestation of the whole representation space.

Note that the construction does not need to change for intransitive group actions. We can still have a fixed starting element x , whose orbit will be used by the minting algorithm to mint banknotes. However, for the proof, we can no longer assume that the two registers produced by the adversary have support on the same orbit—the orbit of x . The adversary may in general attempt to mint banknotes with supports on *different orbits*.

We comment on how the security of the previous section generalizes to the intransitive setting.

Intransitive Preaction Security. We modify the definitions of preaction security to the intransitive case. Let $(G, X, *, x)$ be an intransitive group action.

Assumption 3 (Intransitive Preaction Hardness). *Given $x, g*x$, and h for a fixed starting element, $x \in X$, and random $g, h \leftarrow G$, it is hard to output $gh^{-1} * x$. That is, there exists an $\epsilon > 0$ such that for all QPT adversaries, \mathcal{A} ,*

$$\Pr [z = gh^{-1} * x : x \leftarrow X, g, h \leftarrow G, z \leftarrow \mathcal{A}(x, g * x, h)] \leq \frac{1}{|G|} + \epsilon$$

Assumption 4 (Intransitive Preaction Indistinguishability). *It is hard to distinguish whether a preaction has been performed relative to a set of prefixed starting points. Let $\mathcal{O}_1, \dots, \mathcal{O}_m$ be the orbits of the group action and let x_1, \dots, x_m be representatives from each orbit ($x_i \in \mathcal{O}_i$). Let $F_b^{h_1, h_2} : g * x_i \mapsto h_1 g h_2^{-b} * x_i$, for $b \in \{0, 1\}$ and $h_1, h_2 \in G$. Then there exists an $\epsilon > 0$ such that for all QPT adversaries, \mathcal{A} , that make a single query to $F_b^{h_1, h_2}$,*

$$\Pr [b' = b : h_1, h_2 \leftarrow G, b \leftarrow \{0, 1\}, b' \leftarrow \mathcal{A}^{F_b^{h_1, h_2}}] \leq \frac{1}{2} + \epsilon$$

Note that when $b = 0$, the challenger $F_b^{h_1, h_2}$ performs a group action for a random group element h_1 , and when $b = 1$, it performs both a random group action with h_1 and a random group pre-action with h_2 .

Definition 4.18. We say that a group action of group G_λ on set X_λ with starting element x is ϵ -preaction secure if both [Assumption 3](#) and [Assumption 4](#) hold for the group action against any QPT adversary with advantage ϵ . We say that the group action is preaction secure if it is $\text{negl}(\lambda)$ -preaction secure for any negligible function negl .

We also need an additional technical assumption, which says that it is hard to find “bad” orbits.

Assumption 5 (Intractable bad irreps). *We say that a group action has δ -intractable bad irreps if any QPT adversary has probability at most δ of producing an $x \in X$ and an irrep ϱ such that (1) $\dim(\varrho) > 1$, but (2) ϱ only has a single manifestation in the representation of G acting on the orbit \mathcal{O}_i containing x .*

Note that if all orbits \mathcal{O}_i are in bijection with G , then the representation of G acting on \mathcal{O}_i will have $\dim(\varrho)$ manifestations of each irrep ϱ . However, if some orbit contains an element x such that $g * x = x$ for some g , then the number of manifestations of ϱ may be smaller. [Assumption 5](#) says that it is hard to find such an irrep and representative of such an orbit.

Proposition 4.19. *Suppose that the group action used in the quantum money construction ([Section 4.3.1](#)) is ϵ -intransitive preaction secure and has δ -intractable bad irreps. Then no QPT adversary can produce a quantum state on two registers such the probability of measuring both in the same irreducible subspace is greater than $2\epsilon + \delta$.*

Proof. First, assume that the adversary does not sample a state in an intractable bad orbit. Since the probability is upper bounded by δ , this increase the probability that they measure a state in the same irreducible subspace by δ .

Similar to the proof of [Proposition 4.17](#), we assume for the sake of contradiction that the adversary has an δ probability of measuring two states in the same irreducible representation. Then we consider the same algorithm, [Algorithm 5](#), for distinguishing a black-box that performs a pre-action from a black-box that does not perform a pre-action.

First, let $|\$_{i,j}^{\varrho,x}\rangle$ be the money state that corresponds to irrep label ϱ , manifestation i , basis vector j , and starting element x . Further let $|\phi_i^{\varrho,x}\rangle$ be the archetype state corresponding to irrep ϱ , manifestation i and starting element x .

Case 1: $b = 0$ (there is no preaction). We begin by analyzing the performance of [Algorithm 5](#) in the case when $b = 0$, first in the case where the symmetric subspace accepts and then the case when it fails. Then we can write every state in the symmetric subspace as follows for some choice of α_i^x and $\beta_{j,\ell}$.

$$\sum_{i,k,x,y} \alpha_i^x \alpha_k^y |\phi_i^{\varrho,x}\rangle \otimes |\phi_k^{\varrho,y}\rangle \otimes \sum_{j,\ell} \beta_{j,\ell} |\varrho, \varrho, j, \ell\rangle .$$

Here we note that this encompasses the case when the states span multiple orbits (indexed by starting elements x and y). Then after inverting the Fourier subspace extraction, we get the following state

$$\sum_{i,k,j,\ell,x,y} \alpha_i^x \alpha_j^y \beta_{j,\ell} |\$_{i,j}^{\varrho,x}\rangle \otimes |\$_{k,\ell}^{\varrho,y}\rangle .$$

Then after applying the black box (recall that $b = 0$) to the first register, we have the following

$$\sum_{i,k,j,\ell,x,y} \sum_s \alpha_i^x \alpha_j^y \beta_{j,\ell} \varrho(h_1)_{s,j} |\$_{i,s}^{\varrho,x}\rangle \otimes |\$_{k,\ell}^{\varrho,y}\rangle .$$

After performing Fourier subspace extraction again, we get the following state, following the logic in [Proposition 4.17](#).

$$\sum_{i,k,x,y} \alpha_i^x \alpha_k^y |\phi_i^{\varrho,x}\rangle \otimes |\phi_j^{\varrho,y}\rangle \otimes \sum_{s,\ell,j} (\beta_{j,\ell} \varrho(h_1)_{s,j}) |\varrho, \varrho, s, \ell\rangle .$$

Thus, we measure a state in the symmetric subspace. Furthermore, since the symmetric subspace is perfectly mapped back to the symmetric subspace under the black box, if the first symmetric

subspace measurement outputs the anti-symmetric subspace, the black box will keep the state in the anti-symmetric subspace. Thus, in the case that the $b = 0$, the algorithm accepts with probability 1.

Case 2: $b = 1$ (there is a preaction). We first start in the case when the symmetric subspace projector accepts. In this case, we can write the state after the projector accepts, similarly to before as

$$\sum_{i,k,x,y} \alpha_i^x \alpha_k^y |\phi_i^{\varrho,x}\rangle \otimes |\phi_k^{\varrho,y}\rangle \otimes \sum_{j,\ell} \beta_{j,\ell} |\varrho, \varrho, j, \ell\rangle.$$

After applying the black box (this time with a pre-action), we have the following state

$$\begin{aligned} & \sum_{i,k,x,y,r,s} \alpha_i^x \alpha_k^y \beta_{j,\ell} \varrho(h_2^{-1})_{i,r} \varrho(h_1)_{s,j} |\phi_r^{\varrho,x}\rangle \otimes |\phi_k^{\varrho,y}\rangle \otimes |\varrho, \varrho, s, \ell\rangle \\ &= \sum_{r,k,x,y} \left(\sum_i \alpha_i^x \varrho(h_2^{-1})_{i,r} \right) \alpha_k^y |\phi_r^{\varrho,x}\rangle \otimes |\phi_k^{\varrho,y}\rangle \otimes \sum_{s,\ell} \left(\sum_j \beta_{j,\ell} \varrho(h_1)_{s,j} \right) |\varrho, \varrho, s, \ell\rangle \\ & \left(\sum_{r,x} \alpha_r^x |\phi_r^{\varrho,x}\rangle \right) \otimes \left(\sum_{k,y} \alpha_k^y |\phi_k^{\varrho,y}\rangle \right) \otimes \sum_{s,\ell} \left(\sum_j \beta_{j,\ell} \varrho(h_1)_{s,j} \right) |\varrho, \varrho, s, \ell\rangle. \end{aligned}$$

We can then write the expression for the fidelity of this state and the swapped version if the state as follows.

$$F_{\text{SWAP}} = \sum_{r,k,x,y} \left((\alpha_r^x)^\dagger (\alpha_r^x)^\dagger (\alpha_k^y)^\dagger (\alpha_k^y) \right)$$

Here the only difference from before is that the inner product also enforces that the orbits (x and y) are the same between the left and right. Expanding each α' as before, we get the following expression for the fidelity of the state with its swap.

$$\sum_{r,k,x,y} (\alpha_k^y)^\dagger (\alpha_r^x)^\dagger \sum_{i,i'} (\alpha_i^x)^\dagger \alpha_{i'}^y \varrho(h_2^{-1})_{i,r}^\dagger \varrho(h_2^{-1})_{i',k}.$$

Computing the average over the group and applying [Lemma 2.18](#), we get the following quantity

$$\begin{aligned} & \sum_{r,k,x,y} (\alpha_k^y)^\dagger (\alpha_r^x)^\dagger \sum_{i,i'} (\alpha_i^x)^\dagger \alpha_{i'}^y \mathbb{E}_{h_2 \in G} \left[\varrho(h_2^{-1})_{i,r}^\dagger \varrho(h_2^{-1})_{i',k} \right] \\ &= \frac{1}{\dim(\varrho)} \sum_{x,y} \left(\sum_r (\alpha_r^x)^\dagger (\alpha_r^y) \right) \left(\sum_i (\alpha_i^x)^\dagger (\alpha_i^y) \right) \\ &\leq \frac{1}{\dim(\varrho)} \sqrt{\sum_{x,y} \left(\sum_r (\alpha_r^x)^\dagger (\alpha_r^y) \right) \cdot \sum_{x,y} \left(\sum_i (\alpha_i^x)^\dagger (\alpha_i^y) \right)} \\ &= \frac{1}{\dim(\varrho)}. \end{aligned}$$

Here after applying the Schur orthogonality rules, we apply Cauchy-Schwarz and then use the fact that both terms in the square roots are the norm of the original vector, so they are 1. To complete the proof, we note that the probability that the swap test succeeds is given by

$$\frac{1}{2}(1 + F_{\text{SWAP}}) = \frac{1}{2} + \frac{1}{2 \dim(\varrho)}.$$

Now we proceed with the analysis in the case that the symmetric subspace measurements outputs the anti-symmetric subspace. We can similarly write the anti-symmetric subspace on the first register as the span of the following vectors (and analyzing the action of the rest of the algorithm on those vectors will imply the action on every state in the anti-symmetric subspace).

$$\frac{1}{\sqrt{2}} \left(|\$\!_{i,j}^{\varrho,x}\rangle \otimes |\$\!_{k,\ell}^{\varrho,y}\rangle - |\$\!_{k,j}^{\varrho,y}\rangle \otimes |\$\!_{i,\ell}^{\varrho,x}\rangle \right).$$

Here we require that $\delta_{x,y}\delta_{i,k} = 0$ (i.e. that at least one of the pairs is different). Similar to before we can write out the state after applying the black box (now with a pre-action), and then the Fourier subspace extraction as follows

$$\begin{aligned} |\psi_{i,j,k,\ell}^{x,y}\rangle &= \sqrt{12} \sum_{r,s} \left(\varrho(h_2^{-1})_{i,r} \varrho(h_1)_{s,j} |\phi_r^{\varrho,x}\rangle \otimes |\phi_k^{\varrho,x}\rangle - \varrho(h_2^{-1})_{k,r} \varrho(h_1)_{s,j} |\phi_r^{\varrho,y}\rangle \otimes |\phi_i^{\varrho,x}\rangle \right) \otimes |\varrho, \varrho, s, \ell\rangle \\ &= \frac{1}{\sqrt{2}} \sum_r \left(\varrho(h_2^{-1})_{i,r} |\phi_r^{\varrho,x}\rangle \otimes |\phi_k^{\varrho,y}\rangle - \varrho(h_2^{-1})_{k,r} |\phi_r^{\varrho,y}\rangle \otimes |\phi_i^{\varrho,x}\rangle \right) \otimes \sum_s \varrho(h_1)_{s,j} |\varrho\varrho, s, \ell\rangle. \end{aligned}$$

Now, in a same fashion as before we can write every state in the anti-symmetric subspace as a linear combination of these basis vectors as $\sum_{i,j,k,\ell,x,y} \alpha_{i,j,k,\ell}^{x,y} |\psi_{i,j,k,\ell}^{x,y}\rangle$. We then need to compute the inner product between this state and the state after swapping with itself, averaged over all group elements. We get the following

$$\begin{aligned} &\mathbb{E}_{h_2 \in G} \left[\sum_{\substack{i,j,k,\ell,x,y \\ i',j',k',\ell',x',y'}} \left(\alpha_{i,j,k,\ell}^{x,y} \right)^\dagger \left(\alpha_{i',j',k',\ell'}^{x',y'} \right) \langle \psi_{i,j,k,\ell}^{x,y} | \text{SWAP} | \psi_{i',j',k',\ell'}^{x',y'} \rangle \right] \\ &= \mathbb{E}_{h_2 \in G} \left[\frac{1}{2} \sum_{x,y,x',y'} \sum_{i,k,i',k'} \sum_{r,r'} \left(\varrho(h_2^{-1})_{i',r}^* \langle \phi_r^{\varrho,x} | \otimes \langle \phi_k^{\varrho,y} | - \varrho(h_2^{-1})_{k',r}^* \langle \phi_r^{\varrho,y} | \otimes \langle \phi_i^{\varrho,x} | \right) \right. \\ &\quad \left. \left(\varrho(h_2^{-1})_{i,r'} |\phi_k^{\varrho,y'}\rangle \otimes |\phi_{r'}^{\varrho,x'}\rangle - \varrho(h_2^{-1})_{k,r'} |\phi_i^{\varrho,x'}\rangle \otimes |\phi_{r'}^{\varrho,y'}\rangle \right) \left(\sum_{j,\ell} \alpha_{i,j,k,\ell}^{x,y} \left(\alpha_{i',j',k',\ell'}^{x',y'} \right)^\dagger \right) \right] \\ &= \mathbb{E}_{h_2 \in G} \left[\frac{1}{2} \sum_{x,y,x',y'} \sum_{i,k,i',k'} \sum_{r,r'} \left(\varrho(h_2^{-1})_{i',r}^* \varrho(h_2^{-1})_{i,r'} \langle \phi_r^{\varrho,x} | \phi_k^{\varrho,y'} \rangle \langle \phi_{k'}^{\varrho,y} | \phi_{r'}^{\varrho,x'} \rangle \right. \right. \\ &\quad - \varrho(h_2^{-1})_{i',r}^* \varrho(h_2^{-1})_{k,r'} \langle \phi_r^{\varrho,x} | \phi_i^{\varrho,x'} \rangle \langle \phi_{k'}^{\varrho,y} | \phi_{r'}^{\varrho,y'} \rangle \\ &\quad - \varrho(h_2^{-1})_{k',r}^* \varrho(h_2^{-1})_{i,r'} \langle \phi_r^{\varrho,y} | \phi_k^{\varrho,y'} \rangle \langle \phi_i^{\varrho,x} | \phi_{r'}^{\varrho,x'} \rangle \\ &\quad \left. \left. + \varrho(h_2^{-1})_{k',r}^* \varrho(h_2^{-1})_{k,r'} \langle \phi_r^{\varrho,y} | \phi_i^{\varrho,x'} \rangle \langle \phi_{i'}^{\varrho,x} | \phi_{r'}^{\varrho,y'} \rangle \right) \left(\beta_{i,k,k',i'}^{x,y,x',y'} \right) \right] \\ &= \mathbb{E}_{h_2 \in G} \left[\sum_{x,y,x',y'} \sum_{i,k,i',k'} \left((\varrho(h_2^{-1})_{i',k})^* \varrho(h_2^{-1})_{i,k'} \delta_{x,y'} \delta_{y,x'} - \varrho(h_2^{-1})_{i',i}^* \varrho(h_2^{-1})_{k,k'} \delta_{x,x'} \delta_{y,y'} \right. \right. \\ &\quad \left. \left. - \varrho(h_2^{-1})_{k',k}^* \varrho(h_2^{-1})_{i,i'} \delta_{x,x'} \delta_{y,y'} + \varrho(h_2^{-1})_{k',i}^* \varrho(h_2^{-1})_{k,i'} \delta_{x,y'} \delta_{y,x'} \right) \left(\beta_{i,k,i',k'}^{x,y,x',y'} \right) \right] \\ &= \sum_{x,y} \sum_{i,k} \frac{1}{2} \beta_{i,k,k,i}^{x,y,y,x} \left(\mathbb{E}_{h_2 \in G} \left[|\varrho(h_2^{-1})_{i,k}|^2 + |\varrho(h_2^{-1})_{k,i}|^2 \right] \right). \end{aligned}$$

In the first equality, we note that the swap only affects the first two registers, so the final two must be the same to survive the inner product, as before. Then we use the fact that the inner product of the states $\langle \phi_a^{\rho,x} | \phi_b^{\rho,y} \rangle = \delta_{x,y} \delta_{a,b}$. Finally, we use the fact that in the anti-symmetric states, we can not have both the orbits *and* the manifestations be the same (as noted in the description of the basis states). This allows us to apply the Schur orthogonality relations and remove the two negative terms when we average over the group elements.

Now, we can apply the same equality that we noted before to bound this by $\frac{2}{\dim(\rho)}$, again noting that the $\beta_{i,k,k,i}^{x,y,y,x}$ correspond to a normalized vector. Thus, the probability that the swap test succeeds is bounded from above by the following

$$\frac{1}{2} + \frac{1}{2 \dim(\rho)}.$$

At this point, we have completed the analysis of the probability that the state passes the second test. In particular, when there is no preaction, the pre-action distinguisher outputs “no preaction” with probability 1, and if there is a preaction the distinguisher outputs preaction with probability at least $\frac{1}{2} - \frac{1}{2 \dim(\rho)}$. Thus, if the adversary instead starts with a state that has probability $\dim(\rho)\epsilon / (\dim(\rho) + 1) \leq 2\epsilon$ of being measured in the tensor product of two copies of the irreducible space of ρ that are not intractable bad irreps, they can first measure the state and then apply this distinguisher to break the preaction indistinguishability with probability ϵ . Adding in the probability (δ) that the adversary measures a intractable bad irrep, we get the desired bound. \square

With this proposition, we have shown that the construction of quantum lightning is secure if instantiated with a ϵ -preaction secure group action (as in [Definition 4.18](#)) that has negl -intractable bad irreps. In the next section, we will provide groups that might meet these conditions, providing the first instantiations of quantum lightning in the plain model from plausible assumptions.

4.4 Instantiations

Here, we discuss some plausible instantiations of our quantum money scheme. Our main focus will be on *symmetric* group actions. First, we note that symmetric group actions have a negligible maximum Plancherel measure [[VK85](#)], a necessary condition for having a secure quantum lightning scheme and for our pre-action security assumption to hold. Symmetric groups also admit an efficient quantum Fourier transform [[Bea97](#)], a necessary condition for the efficiency of our protocol. This makes symmetric group actions a natural candidate for instantiating our scheme.

Graph Isomorphism. Given a graph (V, E) with $|V| = n$, the symmetric group S_n acts on (V, E) by permuting the vertices.

Note that the discrete logarithm problem on graphs is exactly the Graph Isomorphism problem. Graph Isomorphism can be solved in (classical) quasi-polynomial time [[Bab16](#)]. However, it is still conceivable that there is no polynomial-time algorithm, giving a plausible candidate group action.

We also would like S_n to act regularly. If (V, E) has a trivial automorphism group, then S_n acts semiregularly on the orbit of (V, E) . “Most” graphs have trivial automorphism groups [[LM17](#)]. Unfortunately, it is in general presumably hard to identify the orbit of a graph (V, E) , since this would solve the graph isomorphism problem for (V, E) . We therefore appeal to our generalization to intransitive group actions in [Section 4.3.3](#). Therefore, even if there are multiple orbits, our security proof still works.

Permuting Matrices. The symmetric group S_n acts on the set of $n \times n$ symmetric matrices via $\sigma * M = \sigma \cdot M \cdot \sigma^T$. That is, permute the rows and columns of M by σ . This is in fact a generalization of the graph isomorphism group action, using the adjacency matrix of the graph.

McEliece Cryptosystem. The McEliece cryptosystem [McE78] contains an implicit group action. Here, we have the symmetric group acting on matrices, though the operation is quite different. Let \mathbb{F} be a field and $m > n$ be integers. Then consider the set $R_{n,m}$ of row-reduced matrices in $\mathbb{F}^{n \times m}$. Then S_m acts on $R_{n,m}$, with $\sigma * M \rightarrow M'$ where M' is the result of:

- First permute the columns of M according to σ .
- Then row-reduce the result.

Note that the McEliece cryptosystem uses the orbit of a specific matrix M that has good error correcting properties. The original proposal in [McE78] is to use binary Goppa codes. This original proposal has so far resisted (quantum) cryptanalysis efforts.

Note that we do not need any specific properties of M , allowing us to use basically any matrix M . Thus, even if the McEliece cryptosystem is broken, we still get a plausible quantum money candidate.

As for regularity, for a sufficiently wide matrix and/or sufficiently large field \mathbb{F} , S_m will act semiregularly on the orbit of “most” matrices, as shown in the lemma below. As with the Graph Isomorphism case, we do not expect to be able to identify the orbits of typical matrices, so we instead appeal to our generalization to non-transitive matrices.

Lemma 4.20. *Let $m \geq 2n + 1$. Consider sampling M such that (1) the left $n \times n$ matrix is the identity, and (2) the right $n \times (m - n)$ matrix is uniform. Then except with probability $p := m^2 n^2 |\mathbb{F}|^{-1} + (m!) |\mathbb{F}|^{-n}$, S_m will act semiregularly on the orbit of M . In particular, if $|\mathbb{F}| = m^{\omega(1)}$, then p is negligible.*

Proof. Fix a permutation $\sigma \in S_m$ other than the identity. We bound the probability that $\sigma * M = M$.

Let us first suppose that the right $n \times (m - n)$ sub-matrix of M contains all distinct and non-zero entries. Since the entries are uniform and independent, this occurs with probability at most $[mn(mn - 1) + mn] |\mathbb{F}|^{-1} = m^2 n^2 |\mathbb{F}|^{-1}$.

Now consider permuting the columns of M according to σ . Denote the result by M' . Let M'' then be the result of row-reducing M' . We now consider three cases:

- Suppose $\sigma(i) = i$ for all $i \leq n$, meaning σ does not permute the first n columns. In this case, $M'' = M'$. Since the columns are distinct by our conditions on M and σ is not the identity, we have that $M' \neq M$. Thus, in this case $\sigma * M \neq M$.
- $\sigma(i) > n$ for all $i > n$. In other words, σ separately permutes the first n entries and permutes the remaining $m - n$ entries. In this case, M' is obtained from M by permuting the right $n \times (m - n)$ sub-matrix, and M'' is obtained from M' by simply permuting some of the rows of M' . In other words, M'' is obtained from M by permuting the rows and columns of the right $n \times (m - n)$ sub-matrix. As long as the entries of this sub-matrix are distinct, any such permutation of rows and columns will not preserve the matrix.
- $\sigma(i) \leq n$ for some $i > n$. In this case, $M'' \neq M'$. Let $D \in \mathbb{F}^{n \times n}$ be the matrix such that $M'' = D^{-1} \cdot M'$. Then we know that D is not the identity.

We now focus on the last case above. If the first n columns of M' are not full rank, then $M'' \neq M$. So suppose that the first n columns of M' are full rank. This means that D is exactly the first n columns of M' . If $M'' = M$, we then have that $D \cdot M = M'$. In other words, if we take the first n columns of M' (which is just a permuted version of M), and multiply this with M , we get exactly M' .

Moreover, since we are in the case $\sigma(i) \leq n$ for some $i > n$, this also means that $\sigma(j) > n$ for some $j \leq n$. Thus at least one of the columns of D came from the right $n \times (m - n)$ sub-matrix of M .

Since $m \geq 2n + 1$, there will be some column i such that $i > n$ and $\sigma(i) > n$. In other words, this column is not among the first n in either M nor in M' . This column is therefore independent of D . Let v denote the vector of elements in this column.

For $M'' = D \cdot M'$ to be equal to M , we need that $D \cdot v$ is among the original columns of M . There are two possibilities:

- $\sigma(i) \neq i$. In this case, let w be the $\sigma(i)$ -th column of M . Then $M'' = M$ implies $D \cdots v = w$. Since v is random and independent of D , w , this occurs with probability $|\mathbb{F}|^{-n}$.
- $\sigma(i) = i$. In this case, $M'' = M$ implies that $D \cdots v = v$. Fix a v such that all the entries of v are non-zero. Since v came from the right sub-matrix of M and we are assuming all the entries there are non-zero, we can assume that v satisfies this property. Now consider sampling D . D contains some columns that are fixed (those that were originally among the first n in M) and some that are random (those that were not among the first n in M). Moreover, at least one of the columns is random, since one of the rows came from the right sub-matrix of M . Since v is non-zero in all positions, in particular it is non-zero in some position corresponding to a random column of D . Thus, $D \cdot v$ is a random vector. The probability $D \cdot v = v$ over the choice of D is therefore at most $|\mathbb{F}|^{-n}$.

Therefore, the probability that there exists some σ such that $\sigma * M = M$ is at most the sum of

- The probability that the right $n \times (m - n)$ sub-matrix contains non-distinct entries, which is upper bounded by $m^2 n^2 |\mathbb{F}|^{-1}$
- For each $\sigma \in S_n$, $|\mathbb{F}|^{-n}$.

Thus, the overall probability that there exists some σ is at most $m^2 n^2 |\mathbb{F}|^{-1} + m! |\mathbb{F}|^{-n}$. □

4.5 Dual-Mode One-way Homomorphisms

In the previous sections, we gave a construction of quantum money/lightning when the group action is easy but its corresponding preaction is hard. In other words, for any group element g , encoded by the group action as $g * x$, we could only act on g from the left to get $hg * x$, but not from the right to get $gh^{-1} * x$. In this section, we explore how the construction of [Section 4.1](#) changes if we explicitly allow acting on the encoded group element from *both sides*. In this case, we have two different but related representations of the same group—one for the action and one for the preaction. One important difference is that this allows verification to recover both of the fine Fourier indices (compare with the hardness of recovering i in [Lemma 4.15](#)).

In fact, we will see that when we allow the encoding to be a *homomorphism*, we get the surprising but useful property that four different notions of security are all identical, including the hardness of worst-case cloning, average cloning, minting a collision (i.e., breaking lightning security), and preparing the specific uniform superposition state corresponding to the trivial irrep.

We begin by giving a useful security definition for one-way homomorphisms:

Definition 4.21. An injective (but not surjective) homomorphism $P : G \rightarrow H$ ³³ is a dual-mode one-way homomorphism if there exists a fooling function $S : G \rightarrow H$ ³⁴ such that they satisfy the following properties:

- **Efficiency:** There is a QPT algorithm to efficiently compute P . There are also efficient QPT algorithms for computing the group operations on G and H .³⁵
- **Statistical Distance:** Let H_P be the image of P and H_S be the image of S . Then H_S is sufficiently far from H_P . Specifically we require that,³⁶

$$\Pr[P(g)S(h) \in H_P \mid g, h \leftarrow G] \leq 1 - \frac{1}{\text{poly}(\lambda)}$$

- **Indistinguishability:** It is hard to distinguish the images of P and S . Formally, for all QPT adversaries A ,

$$\Pr \left[A(h) = b \mid \begin{array}{l} b \leftarrow \{0, 1\} \quad g \leftarrow G \\ h \leftarrow \begin{cases} P(g) & b = 0 \\ S(g) & b = 1 \end{cases} \end{array} \right] \leq \frac{1}{2} + \text{negl}(\lambda)$$

- **Inaccessibility:** It is hard to sample an element of $H \setminus H_P$. Formally, for all QPT adversaries A ,

$$\Pr \left[h \in H \setminus H_P \mid h \leftarrow A(1^\lambda) \right] \leq \text{negl}(\lambda)$$

Remark 4.22. Note that while we do not explicitly require P to be one-way, this is implied by the definition: any adversary for inverting P can be used to break the indistinguishability security.

Remark 4.23. Combined with statistical distance, inaccessibility provides that the fooling function S is hard to compute even in the forward direction. In a cryptographic implementation we would sample a key pair of a public key pk and secret key sk , which would allow computing P and S , respectively (though we omit this in the definition for generality and simplicity). In other words, in the security game, S is a “secret” function that only the challenger knows. This is why we call it a dual-mode one-way homomorphism: there is a public mode P that is publicly computable, and a private mode S that is only privately computable but mimics P to the public.

³³Technically, it is a family of homomorphisms, $\{P_\lambda : G_\lambda \rightarrow H_\lambda\}_{\lambda \in \mathbb{N}}$ but we omit the security parameter in the notation for succinctness.

³⁴We refer to P and S as the “public” and “secret” transformations, respectively. S may itself be a homomorphism but it need not be. Notice that we do not require S to be efficiently computable.

³⁵In general, the algorithms for computing P and the group operations need only be approximately correct. Moreover, because of the inaccessibility condition, we only need the algorithm for group operations on H to be correct and homomorphic on the image of P .

³⁶This condition prevents P being a fooling function for itself. Note that it is equivalent to the condition that $\Pr[S(h) \notin H_P \mid h \leftarrow G] \geq \frac{1}{\text{poly}(\lambda)}$. Or in other words, that $|H_S \setminus H_P|/|H_S| \geq \frac{1}{\text{poly}(\lambda)}$ in the case where S is injective (which it need not be). The reason we prefer to write the statistical distance condition in this way is that if the promised algorithm, A , for group operations on H is randomized, we can take the probability to also be over this randomness.

$$\Pr[u \in H_P \mid u \leftarrow A(P(g), S(h)), g, h \leftarrow G] \leq 1 - \frac{1}{\text{poly}(\lambda)}$$

Observation 4.24. *We can build a plausible candidate dual-mode one-way homomorphism from any group action for which the computational Diffie-Hellman problem (CDH) is easy but the Discrete Logarithm problem (DLog) is hard.³⁷ Note that cryptographers typically would like CDH to be hard, since it allows for justifying the security of more varied protocols. Our construction therefore gives a “win-win” result, showing that in any group action for which DLog is hard, either (1) the more useful CDH problem is actually also hard, or (2) we obtain a plausible candidate dual-mode one-way homomorphism and hence a plausible quantum lightning scheme based on DLog. This win-win result is reminiscent of win-win results in [Zha21], though the details are entirely different.*

Main idea. We give the informal idea here but we leave a formal construction of dual-mode one-way homomorphisms from group actions to future work. Suppose we have a group G which acts on set X . Assume that the CDH problem is easy, and let A be the CDH adversary, which takes two elements $a * x, b * y \in X$ and outputs $ab * x$ if $x = y$, and behaves arbitrarily if $x \neq y$. We set H to be the set X with the “group operation” defined by A .³⁸ Let $x, y \in X$ be two set elements in different orbits of the group G , and let $P(g) = g * x$ and $S(g) = g * y$. Statistical distance comes from the fact that x and y are in different orbits.³⁹ Indistinguishability would come from the hardness of deciding if two elements are in the same orbit (one example being the hardness of the graph isomorphism problem). Inaccessibility would arise from the hardness of sampling a valid set element outside the orbit of any known elements.⁴⁰ Although we argue that these are reasonable assumptions, we do not know if any specific instantiations of group actions satisfy these requirements. We leave finding concrete instantiations of dual-mode one-way homomorphisms to future work. \square

The lack of concrete instantiation of a dual-mode one-way homomorphism is a certainly disadvantage (as opposed to our construction from preaction security in Section 4.3.1, to which we give concrete candidate instantiations in Section 4.4). However, we believe that our construction from dual-mode one-way homomorphism is interesting in its own right. For instance, we have the property that four different security definitions—including average-case and worst-case cloning, as well as quantum lightning security—are all equivalent. To the best of our knowledge, this is the first plausible quantum money construction to have this useful property.

4.5.1 Quantum Money Construction

Let G be a group satisfying the requirements in Section 4.1, and let (P, S) be a dual-mode one-way homomorphism from G to H . We build a quantum lightning scheme, $(\text{Mint}, \text{Ver})$ as follows:

$\text{Mint}(1^\lambda) \rightarrow (\sigma, |\$\sigma\rangle)$: Consider the group action of G on H that comes from left-multiplying an element $h \in H$ by the image of a group element $g \in G$ under P , with the starting element $x \in H$ being the identity element of H . That is, we have the group action $g * y \mapsto P(g)y$ for any $y \in H$.

³⁷Note that such a group action is only possible for non-Abelian groups, since CDH and DLog are known to be computationally equivalent for Abelian groups [MZ22], further demonstrating the necessity of non-Abelian-ness in our generalizations.

³⁸Since the adversary may act arbitrarily when $x \neq y$, this is not exactly a group operation. That is, it only defines a group operation on elements within the same orbit, but this will be sufficient for our purposes.

³⁹Depending on the CDH adversary, the roles of x and y may need to be reversed in order to formally satisfy the statistical distance property. If the CDH adversary, when run on a random element of the orbit of x and a random element of the orbit of y , is more likely to produce an element of the orbit of x then we switch the roles. In other words, we set $P(g) = g * y$ and $S(g) = g * x$. In any case, one of the two choices suffices.

⁴⁰This can be argued in the generic group model [Zha24].

Observe that because our starting element is the identity of H , we have an efficiently computable preaction as well, by multiplying in the same way on the right.

Minting follows from the construction in [Section 4.1](#), and produces a serial number $\sigma \leftarrow \varrho$ denoting the measured irrep, and quantum money state $|\$\varrho\rangle = \sum_{i,j \in [d_\varrho]} \alpha_{ij} |\$\varrho_{ij}\rangle$, where

$$|\$\varrho_{ij}\rangle := \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{h \in G} \varrho(h^{-1})_{ij} |P(h)\rangle$$

$\text{Ver}(\sigma, |\mathcal{L}\rangle) \rightarrow \{\text{accept, reject}\}$: We follow the framework in [Section 4.1](#) to verify under the two group actions (the action and the preaction) consisting of the left and right group operations on the encoded element. Note that within the image of P , this verification accepts any state of the original minted form, as well as states that are of that form, but that are shifted by an element of the center of G . In the security analysis, we show that these are the only states that pass verification.

4.5.2 Security Analysis

Theorem 4.25. *If (P, S) is a secure dual-mode one-way homomorphism for G_λ , then $(\text{Mint}, \text{Ver})$ is a secure quantum lightning scheme.*

Proof. Let C be an adversary for the quantum lightning scheme, which outputs a state on two registers which both pass verification for the same serial number ϱ . We will show that it can be used to break the dual-mode one-way homomorphism.

Specifically, we will show that we can use C to break either the inaccessibility security or the indistinguishability security of the dual-mode one-way homomorphism.

Claim 4.26. *We can assume without loss of generality that the output of C is a tensor product and that the states both have the form $|\$\varrho_{ij}\rangle$, where*

$$|\$\varrho_{ij}\rangle := \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |P(g)\rangle$$

for some $i, j \in [d_\varrho]$.

Proof. We begin by observing that if the quantum money state had non-negligible support on $H \setminus H_P$, then we can measure to get an element outside of the image of P and break the inaccessibility security of the dual-mode one-way homomorphism. Therefore, up to negligible error, we can assume that they both have support only on the image of P . Furthermore, we can assume that both have the same i and j , since we can perform a Fourier subspace extraction twice on each state—both on the action and on the preaction—to get the corresponding i and j , and then change them to match. This gives us a tensor product state $|\$\varrho_{ij}\rangle \otimes |\$\varrho_{ij}\rangle$. \square

We now show how to break indistinguishability security. We consider one of the copies, setting aside the other copy for now.

Suppose we get as input an element $z \in H$ that is either a in the image of P , that is $z = P(h)$, or the image of S , $z = S(h)$, for group element $h \in G_\lambda$. We left-multiply the quantum money state

by z . If it is in the image of P , then we get

$$\begin{aligned}
z \cdot |\$\!_{ij}^{\varrho}\rangle &= \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |P(h)P(g)\rangle \\
&= \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |P(hg)\rangle \\
&= \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1}h)_{ij} |P(g)\rangle \\
&= \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G, k \in [\dim(\varrho)]} \varrho(g^{-1})_{ik} \varrho(h)_{kj} |P(g)\rangle \\
&= \sum_{k \in [\dim(\varrho)]} \varrho(h)_{kj} \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ik} |P(g)\rangle \\
&= \sum_{k \in [\dim(\varrho)]} \varrho(h)_{kj} |\$\!_{ik}^{\varrho}\rangle
\end{aligned}$$

If it is in the image of S , then we similarly get

$$\begin{aligned}
z * |\$\!_{ij}^{\varrho}\rangle &= \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |S(h) * P(g)\rangle \\
&= \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ij} |\tilde{S}(hg)\rangle \\
&= \sum_{k \in [\dim(\varrho)]} \varrho(h)_{kj} \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{ik} |\tilde{S}(g)\rangle
\end{aligned}$$

where \tilde{S} is some function implied by S that is guaranteed by the statistical distance property of [Definition 4.21](#) to have at least inverse polynomial support outside of the image of P .⁴¹

We finally perform a Fourier subspace extraction and swap test with the copy that was set aside. If z was in the image of P , then the swap test will certainly pass. Otherwise, we observed that the two tested states will have orthogonal support that is at least inverse polynomial (since \tilde{S} is far from P), and the swap test will fail with probability $1 - \frac{1}{\text{poly}(\lambda)}$. This therefore breaks the indistinguishability security of the dual-mode one-way homomorphism. \square

4.5.3 Worst-case to Average-case Reduction for Cloning

Remarkably, the problem of cloning any specific (worst-case) money state in this construction can be reduced to that of producing two copies of an average case money state, and therefore to cloning an average-case state. Moreover, all of these are equivalent to the problem of preparing the trivial irrep state (that is, the positive uniform superposition over the image of P).

⁴¹Note that this does not break inaccessibility security, since in this case we are given z which is itself already outside the image of P .

Theorem 4.27 (Worst-case to Average-case Cloning Reduction and Money/Lightning Equivalence). *For the quantum money/lightning scheme defined in Section 4.5.1, the following are equivalent:*

1. *There exists an efficient worst-case cloner that clones all valid money states $|\$\!_{ij}^\varrho\rangle$.*
2. *There exists an efficient average-case cloner that clones an average-case money state $|\$\!_{ij}^\varrho\rangle$, where ϱ is sampled according to the the Plancherel measure of ϱ in the group.*
3. *There exists an efficient lightning adversary that produces two copies of the same money state $|\$\!_{ij}^\varrho\rangle$, where ϱ is sampled according to the the Plancherel measure of ϱ in the group.*
4. *There exists an efficient preparation device that prepares the trivial irrep state $|\$\!^I\rangle$, that is, the positive uniform superposition over image of the homomorphism P .*

In other words, all four tasks (worst-case cloning, average-case cloning, sampling state doublets, and trivial irrep state preparation) are computationally equivalent.

Proof. It can be seen directly that $1 \Rightarrow 2$ (since cloning in the worst case trivially implies doing so in the average case), and that $2 \Rightarrow 3$ (using the Mint function to mint a state and then using the cloner to clone it). So it remains to show that $3 \Rightarrow 4$ and that $4 \Rightarrow 1$. We start by showing that $4 \Rightarrow 1$, and then $3 \Rightarrow 4$ will follow from applying the same process in reverse on the doublet produced by the lightning adversary.

Suppose that we had a quantum money state with irrep label ϱ that we would like to clone:

$$|\$\!_{ij}^\varrho\rangle = \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{h \in G} \varrho(h^{-1})_{ij} |P(h)\rangle$$

We run the trivial irrep state preparation adversary to prepare the positive uniform superposition over the image of P :

$$|\$\!^I\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |P(g)\rangle$$

We left multiply the first register (the money state) by the inverse of the second register (the trivial irrep state), producing

$$\begin{aligned} |\$\!_{ij}^\varrho\rangle \otimes |\$\!^I\rangle &\rightarrow \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{h \in G} \varrho(h^{-1})_{ij} |P(g^{-1}h)\rangle \otimes \frac{1}{\sqrt{|G|}} \sum_{g \in G} |P(g)\rangle \\ &= \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{h \in G} \varrho(h^{-1}g^{-1})_{ij} |P(h)\rangle \otimes \frac{1}{\sqrt{|G|}} \sum_{g \in G} |P(g)\rangle \\ &= \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{k \in [\dim(\varrho)]} \sum_{h \in G} \varrho(h^{-1})_{ik} \varrho(g^{-1})_{kj} |P(h)\rangle \otimes \frac{1}{\sqrt{|G|}} \sum_{g \in G} |P(g)\rangle \\ &= \frac{1}{\sqrt{\dim(\varrho)}} \sum_{k \in [\dim(\varrho)]} \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{h \in G} \varrho(h^{-1})_{ik} |P(h)\rangle \otimes \sqrt{\frac{\dim(\varrho)}{|G|}} \sum_{g \in G} \varrho(g^{-1})_{kj} |P(g)\rangle \\ &= \frac{1}{\sqrt{\dim(\varrho)}} \sum_{k \in [\dim(\varrho)]} |\$\!_{ik}^\varrho\rangle \otimes |\$\!_{kj}^\varrho\rangle \end{aligned} \tag{9}$$

We now have two states that are both valid quantum money states for irrep label ϱ .

Observation 4.28. *If we would like both registers to be exact copies of the original state in tensor product, we can do that as well.*

Proof of Observation 4.28. We apply a left Fourier measurement on the left register (that is, a Fourier measurement corresponding to left action by plaintext group elements) and a right Fourier measurement on the right register (corresponding to right action), to produce

$$\begin{aligned} & \frac{1}{\dim(\varrho)^{3/2}} \sum_{k,\ell,m \in [\dim(\varrho)]} |\$_{i\ell}^{\varrho}\rangle \otimes |L_{\ell k}^{\varrho}\rangle \otimes |L_{km}^{\varrho}\rangle \otimes |\$_{mj}^{\varrho}\rangle \\ \xrightarrow{\text{QFT}} & \frac{1}{\dim(\varrho)^{3/2}} \sum_{k,\ell,m \in [\dim(\varrho)]} |\$_{i\ell}^{\varrho}\rangle \otimes |\varrho, \ell, k\rangle \otimes |\varrho, k, m\rangle \otimes |\$_{mj}^{\varrho}\rangle \end{aligned}$$

Now note that the registers containing k are in the pure state $\frac{1}{\sqrt{\dim(\varrho)}} \sum_{k \in [\dim(\varrho)]} |k\rangle |k\rangle$, in tensor product with the rest of the state. Replace these registers with the state $|j\rangle |i\rangle$ to get

$$\frac{1}{\dim(\varrho)} \sum_{\ell, m \in [\dim(\varrho)]} |\$_{i\ell}^{\varrho}\rangle \otimes |\varrho, \ell, j\rangle \otimes |\varrho, i, m\rangle \otimes |\$_{mj}^{\varrho}\rangle$$

Now uncompute the two Fourier subspace extractions we have just performed to get $|\$_{ij}^{\varrho}\rangle \otimes |\$_{ij}^{\varrho}\rangle$ as desired. \square

We now continue with the proof of [Theorem 4.27](#) and show that [3](#) \Rightarrow [4](#). Given a doublet pair of quantum money states for the same irrep label, we show how to prepare the trivial irrep state $|\$^I\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |P(g)\rangle$. This doublet produced by the lightning adversary will be a state on two registers that passes verification, of the form⁴²

$$\sum_{i,j,k,\ell \in [\dim(\varrho)]} \alpha_{i,j,k,\ell} |\$_{ij}^{\varrho}\rangle \otimes |\$_{k\ell}^{\varrho}\rangle$$

As above, we can use Fourier subspace extractions to convert this to the state $|\$_{ij}^{\varrho}\rangle \otimes |\$_{ij}^{\varrho}\rangle$ or even to the state $\frac{1}{\sqrt{\dim(\varrho)}} \sum_{k \in [\dim(\varrho)]} |\$_{ik}^{\varrho}\rangle \otimes |\$_{kj}^{\varrho}\rangle$. By reversing the process described above, from [Equation \(9\)](#) backwards, we then recover the state $|\$_{ij}^{\varrho}\rangle \otimes |\$^I\rangle$, where the second register is the trivial irrep state, as desired. \square

So cloning an average-case state is as hard as cloning a worst case state, both of which are as hard as preparing the positive uniform superposition over the image of the homomorphism (the trivial irrep state). Moreover, quantum money security and quantum lightning security are equivalent for this scheme.

Remark 4.29. *Note that a task that is absent from [Theorem 4.27](#) is the ability to prepare any quantum money state given its irrep label (ie. its serial number). An adversary for this task would clearly imply one for all four of the tasks mentioned in [Theorem 4.27](#), but it is not clear if the opposite is true. That is, an adversary that breaks the quantum money/lightning scheme nevertheless might not be able to prepare specific money states on command (only at random). In [Section 5](#), we take advantage of precisely this gap to propose a new quantum cryptographic primitive: quantum fire.*

⁴²This is assuming the inaccessibility security of the dual-mode one-way homomorphism, which is what prevents the state from having non-negligible support outside the image of P .

5 Quantum Fire: Quantum States that are Clonable but Untelegraphable

In this section, we introduce a new quantum cryptographic primitive, “quantum fire”, a cryptographic version of the clonable-but-untelegraphable states introduced by [NZ23]. Much like fire is an entropic state of matter that is hard to spark on command, but easy to spread around as long as it is kept alive, *quantum fire* is a quantum state that is hard to prepare but easy to clone as long as it is maintained in coherent quantum form. More specifically, a *quantum fire state*, $|\phi_i\rangle$, comes from a collection $\{|\phi_i\rangle\}_i$ of states that

- **Efficiently sparked:** there is an efficient sparking algorithm that outputs a random $|\phi_i\rangle$, from some distribution over i ,
- **Efficiently clonable:** there is an efficient cloner that maps one copy of $|\phi_i\rangle$ to two copies,
- **Un-telegraphable:** no efficient adversary can encode $|\phi_i\rangle$ into a classical string that can later be revived back into $|\phi_i\rangle$.

We also allow an efficiently verifiable version, in which we have the additional property,

- **Verifiable:** there is a verification algorithm that takes a label i as well as a claimed state $|\phi'_i\rangle$, and outputs whether $|\phi'_i\rangle$ is a valid quantum fire state.⁴³

Quantum fire has been demonstrated to exist relative to a quantum oracle in [NZ23]. However, until now, no plausible construction in the plain model was known. Even the task of finding quantum states that are efficiently clonable without an oracle—but not trivial enough to be described classically—has been a challenge. We give the first candidate construction of quantum fire in the plain model. We challenge the cryptographic community to find either a proof of its security from reasonable assumptions or to break it. We further challenge the community to find and propose other reasonable candidate constructions for quantum fire. Much like the 15-year challenge of finding reasonable constructions of public-key quantum money has led to a variety of new techniques for proving unclonability, we expect the task of finding candidate quantum fire constructions to prove to be a challenging task and to require new and specialized techniques for showing untelegraphability.

5.1 Definition

Quantum fire was implicit in the oracle construction of [NZ23], but no official definition was given. We give a definition of quantum fire as follows:

Definition 5.1 (Quantum Fire). *A quantum fire scheme consists of four quantum algorithms $\mathcal{S} = (\text{KeyGen}, \text{Spark}, \text{Clone}, \text{Ver})$ where*

- $\text{KeyGen}(1^\lambda)$ takes as input the security parameter 1^λ and outputs a private/public key pair (sk, pk) ,
- $\text{Spark}(\text{pk})$ takes the public key and outputs a serial number s and a quantum fire state $|\phi^s\rangle$, which we refer to as a flame,

⁴³In the general case, the verification algorithm for the quantum fire state may allow a larger space of states than those that would be produced by the sparking algorithm. In this case the cloning algorithm and the telegraphing adversary must output any state(s) that pass verification.

- $\text{Clone}(\text{pk}, s, |\phi^s\rangle)$ takes as input the public key pk , a serial number s , and a flame $|\phi^s\rangle$, and outputs two registers AB in some potentially entangled state σ_{AB}^s ,
- $\text{Ver}(\text{pk}, s, \sigma)$ takes as input the public key pk , a serial number s , and an alleged flame σ , and either accepts or rejects.⁴⁴

A quantum fire scheme \mathcal{S} satisfies correctness if for all λ , sparking is correct

$$\Pr \left[\text{Ver}(\text{pk}, s, |\phi^s\rangle) \text{ accepts} \mid \begin{array}{l} (\text{sk}, \text{pk}) \leftarrow \text{KeyGen}(1^\lambda) \\ (s, |\phi^s\rangle) \leftarrow \text{Spark}(\text{pk}) \end{array} \right] \geq 1 - \text{negl}(\lambda),$$

and cloning is correct

$$\Pr \left[\text{Ver}(\text{pk}, s, \cdot) \text{ accepts both registers of } \sigma_{\text{AB}}^s \mid \begin{array}{l} (\text{sk}, \text{pk}) \leftarrow \text{KeyGen}(1^\lambda) \\ (s, |\phi^s\rangle) \leftarrow \text{Spark}(\text{pk}) \\ \sigma_{\text{AB}}^s \leftarrow \text{Clone}(\text{pk}, s, |\phi^s\rangle) \end{array} \right] \geq 1 - \text{negl}(\lambda).$$

Untelegraphability [NZ23] of quantum fire means that it is hard to encode a flame as a classical encoding which can later be brought back. That is, once the flame is extinguished, it is gone. We model this as a pair of adversaries. The first is tasked with deconstructing the flame into a classical message, and the second must use the deconstructed classical message to reconstruct the state.⁴⁵

Algorithm 6 (Quantum Fire Telegraphing Security Game).

1. Challenger generates $(\text{sk}, \text{pk}) \leftarrow \text{KeyGen}(1^\lambda)$, $(s, |\phi^s\rangle) \leftarrow \text{Spark}(\text{sk})$ and send $(\text{pk}, s, |\phi^s\rangle)$ to adversary A .
2. Adversary A returns a classical encoding of the flame $c \in \{0, 1\}^*$.
3. Challenger passes $c \in \{0, 1\}^*$ to adversary B .
4. Adversary B returns a claimed quantum state σ .
5. Challenger runs $\text{Ver}(\text{pk}, s, \sigma)$ and outputs the result.⁴⁶

Definition 5.2 (Quantum fire security). *A quantum fire scheme \mathcal{S} satisfies ϵ -quantum-fire security if for all pairs of efficient adversaries A and B , the success probability of A in the Telegraphing Security Game (Algorithm 6) is at most $\epsilon(\lambda)$.*

We require that Spark and Clone are efficient (QPT) algorithms. Ver may be an efficient public verification, an efficient private verification, or an inefficient verification, leading to three different kinds of quantum fire (publicly verifiable quantum fire, privately verifiable quantum fire, and statistically verifiable quantum fire).

⁴⁴ Ver may not exist for unverifiable quantum fire, or it may require the secret key sk for secretly verifiable quantum fire.

⁴⁵Note that the two adversaries should *not* be entangled, as this allows them to *teleport* the state. Furthermore, maintaining any entanglement implies having to store a quantum register, which is what telegraphing aims to avoid.

⁴⁶In the case of unverifiable quantum fire, the challenger verifies the telegraphing by measuring in a basis containing the valid flame $|\phi^s\rangle$.

5.2 Construction

Let G and H be two groups, and let $f : G \rightarrow H$ be an injective homomorphism between them, which we assume to be one-way. Let H_f be the image of f . Let $\mathcal{F} : G \rightarrow U(\mathcal{H})$ be the representation of G which acts as $\mathcal{F}(g) |h\rangle = |f(g) \cdot h\rangle$. Let the set of quantum fire labels (or serial numbers) be \widehat{G} , the set of irreps of G , and for each $\varrho \in \widehat{G}$, we let valid flames be any state in isotypic component of irrep ϱ , that is, any state of the form

$$|\phi_{ij}^\varrho\rangle = \sum_{g \in G} \varrho(g^{-1})_{ij} |f(g)\rangle$$

We further assume that there is an efficient algorithm to prepare a uniform superposition over the image group H_f : $|\Phi\rangle = \sum_{h \in H_f} |h\rangle$, or a quantum state that approximates it.⁴⁷

Verification and Sparking. Verification is the same as verification for the quantum money construction of [Section 4.1](#): we perform a course Fourier measurement to produce the irrep label ϱ and compare with the claimed quantum fire label. Likewise, to spark a quantum fire state—that is, to prepare a fire state with a random label—run the same verification process on the identity element of H , which samples the irrep label ϱ according to the Plancherel measure of G .

Cloning. We are given a quantum fire state $|\phi_{ij}^\varrho\rangle$ with label ϱ , and we would like to output two such fire states, both of which pass verification for the same label ϱ .

We first prepare $|\Phi\rangle = \sum_{h \in H_f} |h\rangle$, a uniform superposition over the image group H_f . Together with the fire state, we now have the overall state

$$|\phi_{ij}^\varrho\rangle \otimes |\Phi\rangle = \sum_{\substack{g \in G \\ h \in H_f}} \varrho(g^{-1})_{ij} |f(g)\rangle |h\rangle$$

Since f is injective, and therefore bijective between G and H_f , we can reindex the sum over h as

$$= \sum_{g, h \in G} \varrho(g^{-1})_{ij} |f(g)\rangle |f(h)\rangle$$

Both registers contain an element of H . We apply the inverse group operation of the second

⁴⁷Supposedly, this image group is known to all parties, while the specific mapping between G and H_f could be arbitrary.

register into the first register on the left to get

$$\begin{aligned}
&\rightarrow \sum_{g,h \in G} \varrho(g^{-1})_{ij} |f(h)^{-1} \cdot f(g)\rangle |f(h)\rangle \\
&= \sum_{g,h \in G} \varrho(g^{-1})_{ij} |f(h^{-1} \cdot g)\rangle |f(h)\rangle \\
&= \sum_{g,h \in G} \varrho(g^{-1} \cdot h^{-1})_{ij} |f(g)\rangle |f(h)\rangle \\
&= \sum_{\substack{g,h \in G \\ k \in [\dim(\varrho)]}} \varrho(g^{-1})_{ik} \varrho(h^{-1})_{kj} |f(g)\rangle |f(h)\rangle \\
&= \sum_{k \in [\dim(\varrho)]} \sum_{g \in G} \varrho(g^{-1})_{ik} |f(g)\rangle \sum_{h \in G} \varrho(h^{-1})_{kj} |f(h)\rangle \\
&= \sum_{k \in [\dim(\varrho)]} |\phi_{ik}^\varrho\rangle \otimes |\phi_{kj}^\varrho\rangle
\end{aligned}$$

This produces two quantum fire states that both pass verification for the same original label ϱ . While not necessary, if we wish, we could even force the two new fire states to have the same i and j values as the original, and in doing so disentangle them. We simply perform a Fourier subspace extraction on both states from both the left and right side to extract out the new i and j values, replace them with the old i and j , and uncompute⁴⁸ to get the tensor product:

$$|\phi_{ij}^\varrho\rangle \otimes |\phi_{ij}^\varrho\rangle$$

Untelegraphability. We have shown above that these states are efficiently clonable. In order for the construction to be a secure quantum fire scheme, the states must also be *untelegraphable*. That is, there must be no way to deconstruct the states into a classical message that can later be reconstructed back into the quantum state, or at least one that properly passes verification. We leave as an open problem to find sufficient conditions on the one-way homomorphism that would allow showing untelegraphability in the plain model.

Remark 5.3. *The untelegraphability of such a scheme is known to be difficult to prove even relative to a classical oracle: Nehoran and Zhandry [NZ23] show the security of their implicit quantum fire scheme relative to a unitary quantum oracle. Unfortunately, they also show that the same quantum fire construction leads to a unitary oracle separation between the complexity classes clonableQMA and QCMA, and therefore between QMA and QCMA. As generalization of this, they observe that any provably secure and public-key quantum fire scheme relative to a classical oracle will likely lead to a classical oracle separation between QMA and QCMA, an major longstanding open problem of Aharonov and Naveh [AN02] that remains unresolved despite recent progress.*

Observation 5.4. *We observe that while one-wayness may not be a sufficient condition for untelegraphability, is a necessary condition. This is because if we can invert the homomorphism—and we can also perform a quantum Fourier transform on the group—then we can telegraph the state as the classical description of ϱ , i , and j .*

⁴⁸See [Observation 4.28](#) for more details on how to do this.

Proof sketch. Suppose, for instance, that we can invert f perfectly. Then we can do the following. Alice starts with a quantum fire state $|\phi_{ij}^g\rangle = \sum_{g \in G} \varrho(g^{-1})_{ij} |f(g)\rangle$ and inverts f to get

$$\sum_{g \in G} \varrho(g^{-1})_{ij} |f(g)\rangle |g\rangle .$$

She now uncomputes $f(g)$ in the first register to get

$$\sum_{g \in G} \varrho(g^{-1})_{ij} |g\rangle ,$$

which is the left-regular Fourier basis state $|\mathcal{L}_{i,j}^g\rangle$. Taking the quantum Fourier transform of this state then yields $|\varrho\rangle |i\rangle |j\rangle$, which is a classical string that Alice can send to Bob. Bob can then invert this process to recover $|\phi_{ij}^g\rangle$. \square

The notion of quantum fire was featured implicitly in the work of Nehoran and Zhandry [NZ23], where they show that such an object exists relative to a unitary quantum oracle. Their construction uses two oracles: a (quantumly accessible) random oracle, which serves effectively as a verification oracle, and a unitary oracle, which is used for cloning the resulting states. Unfortunately, the scheme of [NZ23] offers little hope of leading to a plain-model instantiation. This is because, as they note, the unitary implemented by the unitary cloning oracle is one that cannot be implemented efficiently.

One approach to strengthen their result is to give a construction from classical functionality. A priori, however, it is not even clear that *any* classical functionality can bestow clonability on a state that cannot be encoded classically. To the best of our knowledge, every known method of efficiently cloning quantum states first passes through the classical description of the states, copies this classical description, and then recovers two clones of the quantum state from the classical descriptions. However, this automatically means that such states are efficiently telegraphable—they can be stored as their classical descriptions. *How can we clone a quantum state (using efficient classical functionality) without ever going through a classical description?*

We answer this question here by giving a proof of concept that this kind of cloning is in fact possible, along with a framework for using it to construct quantum fire with conjectured security. An interesting aspect of our cloning procedure is that the quantum states of the two registers inherently become entangled during the course of the procedure, and only become disentangled at the end. Furthermore, it requires applying a controlled group operation between the two registers. These aspects together give intuitive justification for the untelegraphability of this cloning procedure: controlled operations and more general entangling procedures *cannot* occur over a classical channel.

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